12.802

Small Scale Ocean Dynamics

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Energetics of turbulent flows

Examination of the energy budgets of turbulent flows can provide useful insights into the factors driving the turbulence and leading to its decay. Energy budgets are especially useful to consider in stratified flows such as the atmosphere and oceans, where there is exchange between potential and kinetic energy, and hence changes in the flow can be related to changes in the buoyancy field.

Questions to be asked when examining the energy budgets include:

1. What is the energy source for the turbulence, i.e. what aspects of the large-scale flow/buoyancy field lead to generation of turbulence?

2. Where does the turbulent energy go? Does it feedback on the large-scale, or is it "lost" to molecular processes?

3. Once generated, how is turbulent energy redistributed both spatially, and between different components of the turbulent flow?

To examine the turbulent energy budgets, we make a separation of fields into a large scale component and a small-scale component:

$$u = \overline{u} + u' \tag{1}$$

where

$$\iiint_V u \, \mathrm{d}V = \overline{u} \; ; \; \iiint_V u' \, \mathrm{d}V = 0 \tag{2}$$

where V is the volume over which the spatial averaging takes place. The turbulent flow is arbitrarily defined to be any flow below the averaging scale. Of course this might include linear waves as well as turbulence, and here we will not be able to distinguish between the two. Equivalently the turbulent flow could also be defined as the perturbation from a time-mean flow.

Kinetic Energy Budgets

$$KE = \frac{1}{2}\mathbf{u}\cdot\mathbf{u} \tag{3}$$

$$\text{KE of mean or large-scale flow} = KE_{mean} = \frac{1}{2}\overline{\mathbf{u}}\cdot\overline{\mathbf{u}} \tag{4}$$

$$\text{KE of turbulent or fluctuating flow} = KE_{turb} = \frac{1}{2}\overline{\mathbf{u}'\cdot\mathbf{u}'} \tag{5}$$

KE of large-scale flow

We begin with the Boussinesq equations, to derive equations for the evolution of KE_{mean}. Consider the portion due to each velocity component separately. In the x-direction, multiply the evolution equation for $\overline{u}$ by $\overline{u}$:

$$\frac{\partial}{\partial t}\left(\frac{\overline{u}^2}{2}\right) + \overline{u}\left(\overline{\mathbf{u}}\cdot\nabla\right)\overline{u} + \overline{u}\overline{\left(\mathbf{u}'\cdot\nabla\right)u'} = -\frac{\overline{u}}{\rho_0}\frac{\partial\overline{p}}{\partial x} + \nu\overline{u}\nabla^2\overline{u} + f\overline{u}\overline{v} \tag{6}$$

Using the following relationships:
$\overline{u}(\overline{\mathbf{u}}\cdot\nabla)\overline{u} = \overline{\mathbf{u}}\cdot\nabla(\overline{u}^2/2)$;
$\overline{u}\overline{(\mathbf{u}'\cdot\nabla)u'} = \nabla\cdot(\overline{\mathbf{u}'u'}\overline{u}) - \overline{u'\mathbf{u}'}\cdot\nabla\overline{u}$
and $\overline{u}\nabla^2\overline{u} = \nabla^2(\overline{u}^2/2) - \nabla\overline{u}\cdot\nabla\overline{u}$,
we can rearrange the u-component of the mean KE equation to give:

$$\begin{aligned}\left(\frac{\partial}{\partial t} + \overline{\mathbf{u}}\cdot\nabla\right)\frac{\overline{u}^2}{2} = -\frac{1}{\rho_0}\frac{\partial}{\partial x}(\overline{p}\overline{u}) + \nu\nabla^2(\frac{\overline{u}^2}{2}) - \nabla\cdot(\overline{\mathbf{u}'u'}\overline{u})& \\ -\nu\nabla\overline{u}\cdot\nabla\overline{u} + \overline{\mathbf{u}'u'}\cdot\nabla u& \\ +f\overline{u}\overline{v} + \frac{\overline{p}}{\rho_0}\frac{\partial\overline{u}}{\partial x}&\end{aligned} \tag{7}$$

The first three terms on the right hand side describe redistribution of mean KE within the volume:
$-\frac{1}{\rho_0}\frac{\partial}{\partial x}(\overline{p}\overline{u})$: pressure work
$\nu\nabla^2(\frac{\overline{u}^2}{2})$: transport by viscous stresses
$-\nabla\cdot(\overline{\mathbf{u}'u'}\overline{u})$: transport by Reynolds stresses.

When integrated over a volume with no flux of KE in or out, these terms are zero.

The 4th and 5th terms represent net sources/sinks of mean KE:
$-\nu\nabla\overline{u}\cdot\nabla\overline{u}$: loss of KE to dissipation;
$\overline{\mathbf{u}'u'}\cdot\nabla u$: transfer of mean KE to the fluctuating/turbulent part of the flow.

The 6th and 7th terms represent transfer of kinetic energy from the $\overline{u}$-component of the flow to the $\overline{v}$- and $\overline{w}$- components.

We can write down similar equations for the time-evolution of $\overline{v}^2$ and $\overline{w}^2$:

$$\begin{aligned}\left(\frac{\partial}{\partial t}+\overline{\mathbf{u}}\cdot\nabla\right)\frac{\overline{v}^2}{2} = -\frac{1}{\rho_0}\frac{\partial}{\partial y}(\overline{pv}) + \nu\nabla^2(\frac{\overline{v}^2}{2}) - \nabla\cdot(\overline{\mathbf{u}'v'}\overline{v}) \\ -\nu\nabla\overline{v}\cdot\nabla\overline{v}+\overline{\mathbf{u}'v'}\cdot\nabla v \\ -f\overline{uv}+\frac{\overline{p}}{\rho_0}\frac{\partial\overline{v}}{\partial y}\end{aligned} \tag{8}$$

$$\begin{aligned}\left(\frac{\partial}{\partial t}+\overline{\mathbf{u}}\cdot\nabla\right)\frac{\overline{w}^2}{2} = -\frac{1}{\rho_0}\frac{\partial}{\partial z}(\overline{pw}) + \nu\nabla^2(\frac{\overline{w}^2}{2}) - \nabla\cdot(\overline{\mathbf{u}'w'}\overline{w}) \\ -\nu\nabla\overline{w}\cdot\nabla\overline{w}+\overline{\mathbf{u}'w'}\cdot\nabla w \\ +\overline{w}\overline{b}+\frac{\overline{p}}{\rho_0}\frac{\partial\overline{w}}{\partial z}\end{aligned} \tag{9}$$

The $\overline{v}^2$ equation contains terms analogous to the $\overline{u}^2$ equation, while the $\overline{w}^2$ equation lacks the Coriolis term (since we have assumed Coriolis is aligned with the vertical), but includes a buoyancy term, through which large scale potential energy is converted to kinetic energy.

If we sum these three equations, to obtain the evolution equation for $1/2\overline{\mathbf{u}}\cdot\overline{\mathbf{u}}$, and rewrite it in Einstein notation, we have

$$\begin{aligned}\left(\frac{\partial}{\partial t}+\overline{u}_j\frac{\partial}{\partial x_j}\right)\frac{\overline{u}_i^2}{2} = \frac{\partial}{\partial x_j}\left(-\frac{\overline{p}}{\rho_0}\overline{u}_j\delta_{i,j}+\nu\frac{\partial}{\partial x_j}\frac{\overline{u}_i^2}{2}-\overline{u_j'u_i'}\overline{u}_i\right) \\ -\nu\left(\frac{\partial\overline{u}_i}{\partial x_j}\right)^2+\overline{u_j'u_i'}\frac{\partial}{\partial x_j}\overline{u}_i+\overline{w}\overline{b}\end{aligned} \tag{10}$$

where the first three terms on the right hand side are once again the transport terms: pressure work, transport by viscous stresses and transport by Reynolds stresses. The 4th term is again the dissipation, and the 5th term represents the transfer of kinetic energy between the mean flow and the turbulent fluctuating flow. This term is known as the **shear production** term, since the shear in the mean flow (finite gradients in $\overline{u}_i$) leads to production of turbulent kinetic energy. The final term on the right hand side is the large-scale buoyancy production term.

Note that the terms $\overline{p}/\rho_0 \partial \overline{u}_i/\partial x_i$, which transfer kinetic energy between the different components of the flow $\overline{u}, \overline{v}, \overline{w}$ vanish from the equation for the total, due to the divergence relation $\nabla \cdot \mathbf{u} = 0$. The Coriolis term similarly does not influence the total kinetic energy, but only its transfer between $\overline{u}$ and $\overline{v}$ components.

KE of turbulent flow

To find the evolution equation for the x-component of the turbulent kinetic energy (TKE) multiply $\partial u'/\partial t = \partial u/\partial t - \partial \overline{u}/\partial t$ by u' and take the spatial average:

$$\begin{aligned}\left(\frac{\partial}{\partial t} + \overline{\mathbf{u}} \cdot \nabla\right) \frac{\overline{u'^2}}{2} = -\frac{1}{\rho_0}\frac{\partial}{\partial x}\overline{u'p'} + \nu \nabla^2 \frac{\overline{u'^2}}{2} - \nabla \cdot \left(\frac{\overline{\mathbf{u}' u'^2}}{2}\right) \\ -\nu \nabla u' \cdot \nabla u' - \overline{u'\mathbf{u}'} \cdot \nabla \overline{u} \\ + f\overline{u'v'} + \frac{1}{\rho_0}\overline{p'\frac{\partial u'}{\partial x}}\end{aligned} \tag{11}$$

Comparing with the equation for $\overline{u}^2/2$ we see that once again, there are three transport terms: pressure work, transport by viscous stresses and transport by Reynolds stresses, and loss of TKE to dissipation. The shear production terms appears once again, but with the opposite sign to that in Eq. 7 - hence this term represents no net loss of KE but a transfer between mean and turbulent components.

The analogous equations for $\overline{v'^2}$ and $\overline{w'^2}$ are:

$$\begin{aligned}\left(\frac{\partial}{\partial t} + \overline{\mathbf{u}} \cdot \nabla\right) \frac{\overline{v'^2}}{2} = -\frac{1}{\rho_0}\frac{\partial}{\partial y}\overline{v'p'} + \nu \nabla^2 \frac{\overline{v'^2}}{2} - \nabla \cdot \left(\frac{\overline{\mathbf{u}' v'^2}}{2}\right) \\ -\nu \overline{\nabla v' \cdot \nabla v'} - \overline{v'\mathbf{u}'} \cdot \nabla \overline{v} \\ - f\overline{u'v'} + \frac{1}{\rho_0}\overline{p'\frac{\partial v'}{\partial y}}\end{aligned} \tag{12}$$

$$\begin{aligned}\left(\frac{\partial}{\partial t} + \overline{\mathbf{u}} \cdot \nabla\right) \frac{\overline{w'^2}}{2} = -\frac{1}{\rho_0}\frac{\partial}{\partial z}\overline{w'p'} + \nu \nabla^2 \frac{\overline{w'^2}}{2} - \nabla \cdot \left(\frac{\overline{\mathbf{u}' w'^2}}{2}\right) \\ -\nu \overline{\nabla w' \cdot \nabla w'} - \overline{w'\mathbf{u}'} \cdot \nabla \overline{w} \\ + \overline{w'b'} + \frac{1}{\rho_0}\overline{p'\frac{\partial w'}{\partial z}}\end{aligned} \tag{13}$$

The $\overline{w'^2}$ equation contains an additional term: the buoyant production of kinetic energy, representing conversion from potential to kinetic energy.

Adding the contributions due to the 3 velocity components and rewriting in Einstein notation we have

$$\left(\frac{\partial}{\partial t} + \overline{u}_j \frac{\partial}{\partial x_j}\right) \frac{\overline{u_i'^2}}{2} = \frac{\partial}{\partial x_j}\left(-\frac{1}{\rho_0}\overline{u_i'p'}\delta_{i,j} + \nu \frac{\partial}{\partial x_j}\frac{1}{2}\overline{u_i'^2} - \frac{1}{2}\overline{u_j'u_i'u_i'}\right)$$

$$-\nu\overline{\left(\frac{\partial u_i'}{\partial x_j}\right)^2} - \overline{u_j'u_i'}\frac{\partial}{\partial x_j}\overline{u}_i + \overline{b'w'} \tag{14}$$

Hence TKE is generated by (a) shear production,

$$P = -\overline{u_j'u_i'}\frac{\partial}{\partial x_j}\overline{u}_i \tag{15}$$

and (b) buoyant production

$$B = \overline{b'w'} \tag{16}$$

and lost through dissipation

$$\epsilon = \nu\overline{\left(\frac{\partial u_i'}{\partial x_j}\right)^2} \tag{17}$$

The buoyant production term may be either positive (generation of kinetic energy, loss of potential energy) or negative (loss of KE, increase in PE).

Now we see the importance of turbulence to the total energy of the system. The viscous terms in the KE of the mean flow can be quite small, so that most KE loss from the mean flow might be due to transfer to the turbulence via the shear production term. Then once in the turbulent regime the KE may either be dissipated, or converted to potential energy via the buoyancy term.

Note that the TKE equations are far from isotropic. Shear production reflects any isotropy in the mean flow, while buoyant production appears only in the $\overline{w'^2}$ equation. The pressure interaction terms (and coriolis terms) transfer the TKE between different velocity components.

If (a) the turbulence is stationary ($DKE_{turb}/Dt = 0$), and (b) we integrate over a volume bounded by surfaces through which there are no energy fluxes, then there is a balance between production and dissipation of TKE:

$$P + B = \epsilon \tag{18}$$

3D homogeneous, isotropic turbulence

G. I. Taylor spearheaded the statistical study of turbulence in a paper published in 1935 in the Proceedings of the Royal Society of London. In that paper he proposed to consider an homogeneous, isotropic turbulent flow. This requires that there is no mean flow (which breaks homegeneity), no rotation and no buoyancy forces (which break isotropy). At equilibrium the turbulent kinetic energy budget would reduce to $\epsilon = 0$. Taylor proposed to assume that the turbulence was supported by an external forcing $\boldsymbol{f}(\boldsymbol{x}, t)$ with no mean. The turbulent kinetic energy takes the form,

$$\overline{\boldsymbol{f}\cdot\boldsymbol{u}} = \epsilon. \tag{19}$$

In 1941 the Russian statistician A. N. Kolmogorov published three papers (in Russian) that provide some of the most important and most-often quoted results of turbulence theory. These results, which will be discussed in some detail in this lecture, comprise what is now referred to as the K41 theory, and represent a milestone of the statistical theory of turbulence. This theory provides a prediction for the energy spectrum of a 3D isotropic homogeneous turbulent flow, i.e. how the energy injected in the flow by the forcing $\boldsymbol{f}$ is redistributed across spatial scales all the way down to the dissipation scale. Kolmogorov proved that even though the velocity of an isotropic homogeneous turbulent flow fluctuates in an unpredictable fashion, the energy spectrum (how much kinetic energy is present on average at a particular scale) is predictable.

The spectral theory of Kolmogorov had a profound impact on the field and it still represents the foundation of many theories of turbulence. It should however be kept in mind that 3D isotropic homogeneous turbulence is an idealization never encountered in nature. The challenge is then to understand what aspects of these theories apply to natural flows and what are pathological.

The notes on 3D turbulence are included in a sperate file.

Pure Shear flow

If the large-scale flow consists of a pure shear flow of the form $(\bar{u}, \bar{v}, \bar{w}) = (U(z), 0, 0)$ with no buoyancy forcing, then the TKE shear production term becomes $\overline{u'w'}\partial \overline{U}/\partial z$, and it appears only in the $\overline{u'^2}$ equation. Hence the large-scale flow directly generates TKE only in the x-direction. $\overline{v'^2}$ and $\overline{w'^2}$ are then generated by transfer of TKE from the x-direction via the pressure interaction terms.

Pure convective flow

If there is no large-scale flow, and turbulence is generated entirely through buoyancy forcing, $P = 0$. The source of TKE is $\overline{w'b'}$, and TKE is directly generated only in the z-direction. Again, $\overline{u'^2}$ and $\overline{v'^2}$ are then generated by transfer of TKE via the pressure interaction terms.

Flux Richardson number

Obviously an important parameter is the ratio between shear and buoyancy production of TKE, known as the flux Richardson number:

$$R_f = \frac{B}{P} = \frac{\overline{w'b'}}{\overline{u'w'}\partial\overline{U}/\partial z} \tag{20}$$

If $\partial U/\partial z > 0$, then $\overline{u'w'} < 0$ if the flux of momentum is downgradient (positive eddy viscosity). Hence we expect $\overline{u'w'}\partial\overline{U}/\partial z < 0$. $R_f < 0$ therefore if $\overline{w'b'} > 0$ (convective instability, buoyancy generating TKE), and $R_f > 0$ if $\overline{w'b'} < 0$ (stable stratification, loss of TKE to PE).

Further reading: Tennekes and Lumley, chapter 3.

Potential Energy Budget

The potential energy is defined as

$$PE = -bz \tag{21}$$

where b is the buoyancy, and z is the vertical distance from some reference level (the obvious choice for ocean and atmospheric scenarios being mean sealevel).

The evolution equation for potential energy is found from

$$\frac{\partial}{\partial t}(-bz) = -z\frac{\partial b}{\partial t} - b\frac{\partial z}{\partial t} = -z\frac{\partial b}{\partial t} - bw \tag{22}$$

Then substituting from the buoyancy evolution equation 1.3, then

$$\left(\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\right)(-bz) = -bw + \kappa\nabla^2(-bz) \tag{23}$$

Now we separate the buoyancy and flow into large and small-scale components, and find the evolution equation for the large-scale potential energy $\overline{-b}z$:

$$\left(\frac{\partial}{\partial t} + \overline{\boldsymbol{u}}\cdot\nabla\right)(-\overline{b}z) = \nabla\cdot\overline{-b'z\mathbf{u}'} + \kappa\nabla\cdot(-z\nabla\overline{b}) + \kappa\frac{\partial\overline{b}}{\partial z} - \overline{b}\overline{w} - \overline{b'w'} \tag{24}$$

The first 2 terms on the right hand side describe transport of PE by Reynolds fluxes and diffusive fluxes. The third term describes conversion of internal to potential energy. The fourth and fifth terms both describe exchange of energy from kinetic energy to potential energy, by the large-scale and small-scale flow components respectively.

Note that the potential energy due to small-scale variations in buoyancy $-b'z$, by definition has zero contribution to the spatially averaged potential energy: $\overline{-b'z} = 0$. The total mean potential energy can therefore be evaluated just by looking at the mean buoyancy profile.

If we combine the equations for the mean and turbulent kinetic energies and the potential energy, and integrate over a volume through which there are no fluxes in or out, we obtain equation for the conservation of total energy.

Tracer variance equation

As for the kinetic energy components, we can derive a time evolution equation for $\overline{T'^2}$ (where T could be any conserved tracer) by multiplying the equation for T' through by T':

$$\left(\frac{\partial}{\partial t} + \overline{\mathbf{u}} \cdot \nabla\right) \frac{\overline{T'^2}}{2} = \kappa \nabla^2 \frac{\overline{T'^2}}{2} - \nabla \cdot \overline{\mathbf{u}' T'^2/2} - \overline{T'\mathbf{u}'} \cdot \nabla \overline{T} - \kappa \overline{\nabla T' \cdot \nabla T'} \tag{25}$$

Like the kinetic energy equation, the first two terms on the right hand side are transport terms (transport by viscous stresses and transport by Reynolds stresses), while tracer variance is produced by the term

$$P_T = -\overline{T'\mathbf{u}'} \cdot \nabla \overline{T} \tag{26}$$

and dissipated by the term

$$\epsilon_T = \kappa \overline{\nabla T' \cdot \nabla T'} \tag{27}$$

If there is a balance between production and dissipation of tracer variance (implying stationarity and homogeneity) then

$$P_T = \epsilon_T \tag{28}$$

Further reading: Tennekes and Lumley Ch 3.

Osborne-Cox model for estimating eddy diffusivity

The Osborne-Cox model relates eddy diffusivities to the observed microstructure in the temperature and velocity fields. It assumes (a) a local balance between production and dissipation (i.e. ignoring transport terms) and (b) a down-gradient turbulent flux of buoyancy or heat.

To calculate an eddy diffusivity for temperature from the temperature microstructure, a production-dissipation balance implies

$$-\overline{T'\mathbf{u}'} \cdot \nabla\overline{T} = \kappa\overline{\nabla T' \cdot \nabla T'} \tag{29}$$

(where κ is the molecular diffusivity). Then if large-scale gradients occur principally in the vertical direction,

$$-\overline{w'T'}\frac{\partial\overline{T}}{\partial z} = \kappa\overline{\nabla T' \cdot \nabla T'} \tag{30}$$

Hence if we can directly measure the components of $\partial T'/\partial x_j$, and the large-scale T gradient, we can estimate the vertical heat flux:

$$\overline{w'T'} = -\kappa\frac{\overline{\nabla T' \cdot \nabla T'}}{\partial\overline{T}/\partial z} \tag{31}$$

This vertical heat flux may be parameterized by a down-gradient eddy diffusion:

$$\overline{w'T'} = -\kappa_T\frac{\partial\overline{T}}{\partial z} \tag{32}$$

so that, by we obtain an estimate for κ_T, the eddy diffusivity:

$$\kappa_T = \kappa\frac{\overline{\nabla T' \cdot \nabla T'}}{(\partial\overline{T}/\partial z)^2} \tag{33}$$

Alternatively, an estimate of κ_b, the eddy diffusivity of buoyancy, can be obtained from the kinetic energy budget.

If production of kinetic energy from the vertical shear of the mean flow is locally balanced by buoyancy production and dissipation (i.e. ignoring transport terms, and assuming stationarity), then

$$\overline{u'w'}\frac{\partial\overline{U}}{\partial z} = \overline{w'b'} - \epsilon \tag{34}$$

Then from the definition of the flux Rayleigh number R_f, $\overline{u'w'}\partial\overline{U}/\partial z = \overline{w'b'}/R_f$. Substituting in the above, we have an estimate of $\overline{w'b'}$:

$$\overline{w'b'} = \frac{R_f}{R_f - 1}\epsilon \tag{35}$$

We can parameterize this buoyancy flux by down-gradient eddy diffusion:

$$\overline{w'b'} = -\kappa_b N^2 \tag{36}$$

where κ_b is the eddy diffusivity of buoyancy, and $N^2 = \partial\bar{b}/\partial z$. Then we have an estimate for κ_b:

$$\kappa_b = \frac{R_f}{1 - R_f} \frac{\epsilon}{N^2} \tag{37}$$

For stratified turbulence, it has been found empirically that $R_f \geq 0.15$. This implies that 15% of the TKE generated from the mean shear goes to increasing the potential energy, while the rest is dissipated. Then if N^2 and ϵ are measured, an upper bound on κ_b can be estimated.

Further reading: Osborn and Cox, 1972: Oceanic fine structure. *Geophys. Fluid Dyn.*, **3**, 321-345.

Osborn, 1980: Estimates of the local rate of vertical diffusion from dissipation measurements. *J. Phys. Oceanogr.*, **10** 83-89.

Winters and D'Asaro, 1996: Diascalar flux and the rate of fluid mixing. JFM v317.