12.802
Small Scale Ocean Dynamics

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Turbulent flows

Richard Feynman (1964) in his *Lecture on Physics* observed that,

> Often, people, in some unjustified fear of physics, say you can't write an equation for life. Well, perhaps we can. As a matter of fact, we very possibly already have the equation to a sufficient approximation, when we write the equation for quantum mechanics:
>
> $$H\psi = -\frac{\hbar}{\mathrm{i}}\frac{\partial \psi}{\partial t}. \tag{1}$$
>
> However, we are unable to reconstruct the field of biology from this equation, and we depend on detailed observation of biological phenomena.

Uriel Frisch (1995) in his book *Turbulence* points out that an analogous situation prevails in the study of turbulent flows. The equations, generally referred to as the Navier-Stokes equations, have been know since Navier (1827) and Stokes (1845),

$$\frac{D\boldsymbol{u}}{Dt} = -\frac{1}{\rho_0}\nabla p + \nu \nabla^2 \boldsymbol{u}, \tag{2}$$

$$\nabla \cdot \boldsymbol{u} = 0, \tag{3}$$

The Navier-Stokes equations probably contain all of turbulence. Yet it would be foolish to try to guess from these equations all the variety of regimes of turbulent

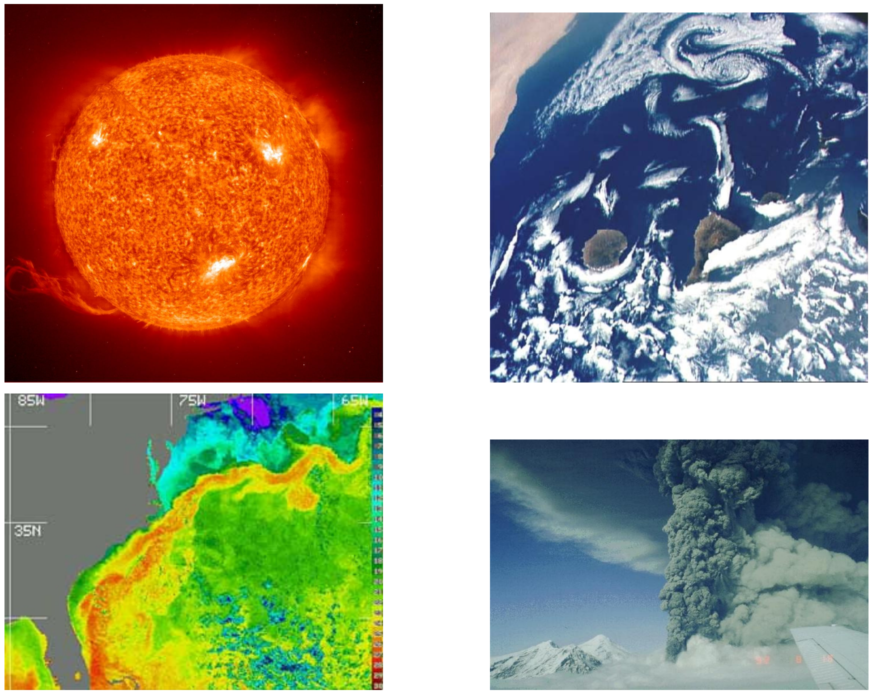


Figure 1: Examples of turbulent flows at the surface of the Sun, in the Earth's atmosphere, in the Gulf Stream at the ocean surface, and in a volcanic eruption.

flows without looking at experimental facts. The phenomena are almost as varied as in the realm of life. The flows shown in Fig. 1 are example of solutions of the Navier-Stokes equations, with modifications to account for rotation and density variations. But nobody knows how to derive these solutions from the equations themselves.

A good way to make contact with the world of turbulence phenomena is through observations of natural flows. Examples are ubiquitous in the ocean, atmosphere, lakes, and rivers of our Earth, in the atmospheres of other planes, in stars, galaxies, and space gases (neutral and ionized). A few examples are shown in Fig. 1. These flows are very irregular and do not display the regularity of the solutions of the Navier-Stokes equations that we studied so far in this course. The field of turbulence can be defined as the attempt to bring together our understanding of the laws that govern fluid dynamics (the Navier-Stokes equations) with the irregular nature of real flows.

But how can we tell which natural flows are turbulent and which are not? As for the problem of defining life, there is no simple answer. A useful approach is to list what properties must be present to consider a flow turbulent.

Properties of turbulent flows

- **Broadband spectrum in space and time**

  Turbulent flows are characterized by structures on a broad range of spatial and temporal scales, even given smooth or periodic initial conditions and forcing. That is turbulent flows have a broadband spectrum both in frequency and wavenumber domains.

  If L is the length scale of the largest motions and l is the length scale of the smallest motions in a flow, then a large range of spatial scales implies $L \gg l$. The scale l is typically the scale at which dissipation becomes important and removes energy from the flow. The scale L, instead, is set by the forcing mechanisms that set the large-scale flow. The ratio L/l is the Reynolds number Re, and $L \gg l$ implies that the Reynolds number be large. Turbulent flows have large Reynolds numbers (Fig. 2 and 3).

- **Dominated by advective nonlinearity**

  A field of non-interacting linear internal waves with many different frequencies and wavenumbers can also have a large range of length scales, but it is not turbulent. Why not? In a turbulent flow the different scales interact, through the nonlinear terms in the equations of motion. And these nonlinear interactions are responsible for the presence of structure on many different scales. Thus the broad band spectrum appears as a result of the internal dynamics. In a field of linear internal waves, instead, the broad band spectrum is generated by external controls like forcing, initial or boundary conditions (Fig. 4).

- **Unpredictable in space and time**

  Turbulent flows are predictable for only short times and short distances. Even though we know the equations that govern the evolution of the fluid, we cannot make predictions about the details of the flow due to its sensitive dependence on initial and boundary conditions. This sensitive dependence is once more a result of the strong nonlinearity of the flow. Predictability, however, can be recovered in a statistical sense, as we will illustrate in a bit (Fig. 5). The sensitive dependence on initial and boundary conditions is a fundamental property of *chaotic* systems. Are thus turbulence and chaos synonyms? No. Turbulent motions are indeed chaotic, but chaotic motions need not be turbulent. Chaos may involve only a small number of degrees of freedom, *i.e.* it can be narrow band in space and/or time. There are numerous examples of chaotic systems characterized by temporal complexity, but spatial simplicity, like the Lorenz's system. Another class of chaotic flows is represented by amplitude equations that describe the slow time and large scale evolution of nearly monochromatic waves. Turbulence is different, because it is always complex both in space and time.

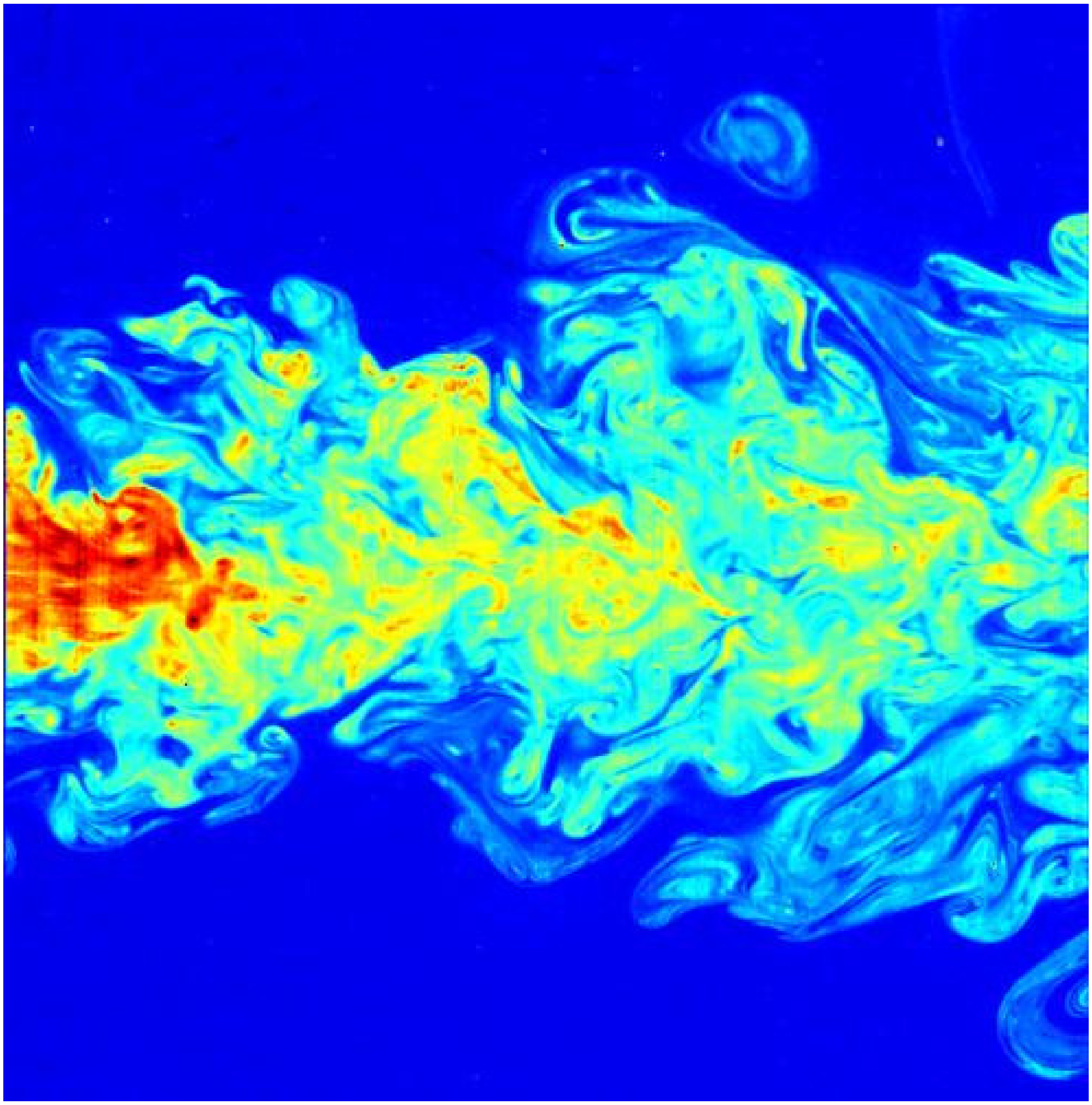
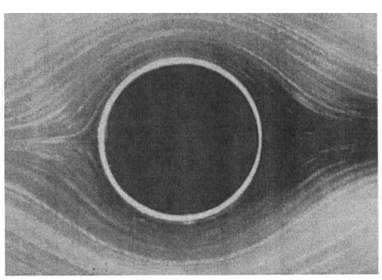


Figure 2: High Re jet exiting from a nozzle.



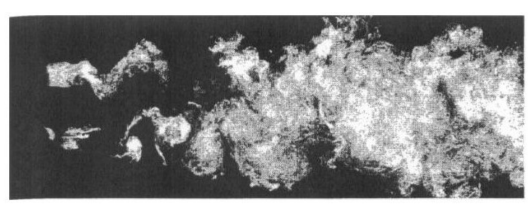


Figure 3: Uniform flow incident on a cylinder at low Re. Uniform flow incident on two cylinders at high Re.

Figure 4: These internal wave photographs were taken by astronauts on board the space shuttle on Jan. 14, 1986. The picture shows the sea surface of the Eastern Pacific, around the Galapagos Islands, 600 miles off the coast of Ecuador. The sea surface coverage of a photograph is about of 75 km by 75 km. There is a clear difference between the wavy patterns of internal waves and the turbulent patterns of clouds.

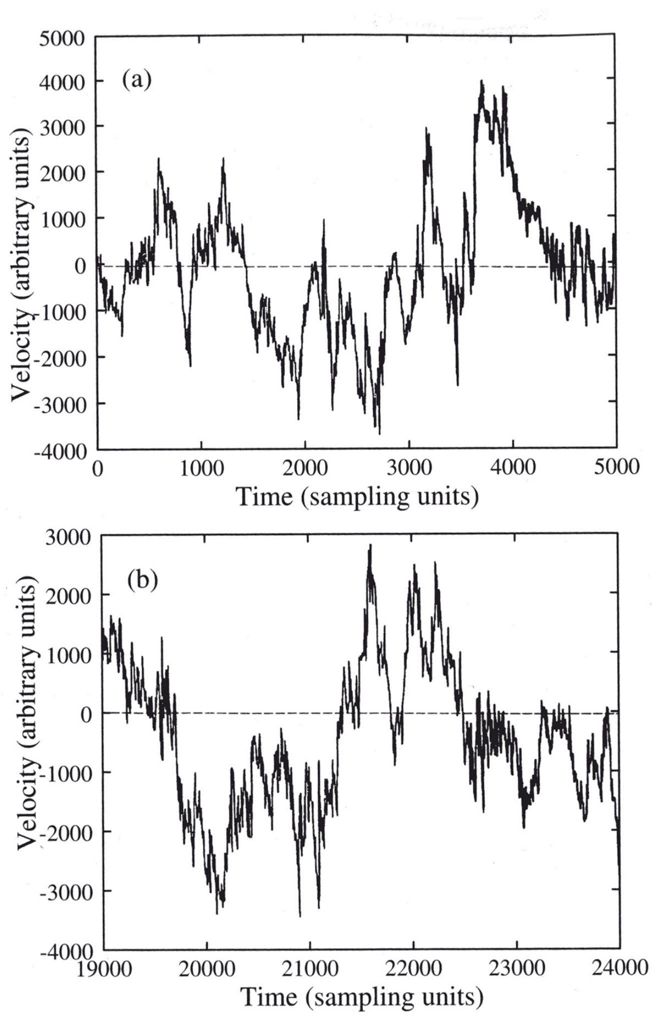


Figure 5: Two sections of one second of a signal recorded by a hot-wire (sampled at 5 kHz) in a wind tunnel. The two sections differ in some small details of the flow upstream, i.e. initial conditions. The statistical properties of the two signals are similar, but the details of the flow are completely different.

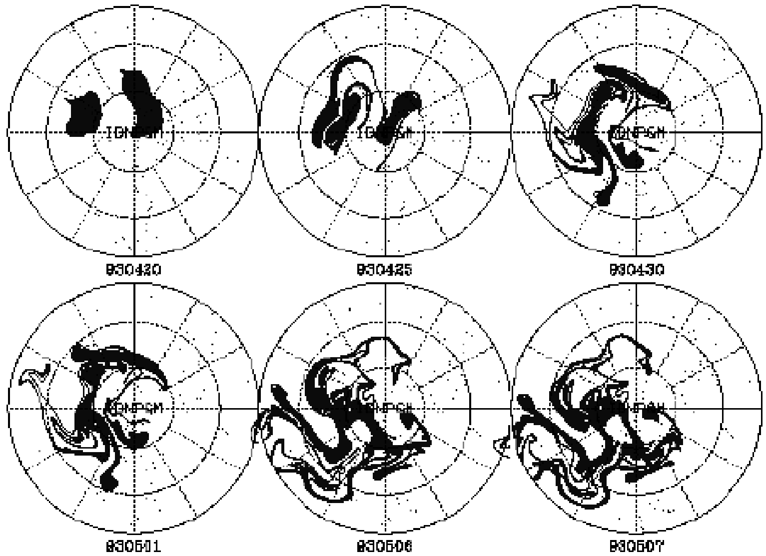


Figure 6: Evolution of a simulated tracer deployed in the Arctic vortex in April 1993. The circles on plots for 30 April, 1, 6, and 7 May are the location where tracer values representative of vortex air where measured aboard the ER-2 aircraft. There is good agreement between the location of the filaments of ex-vortex air in the simulation and the locations where vortex air was observed (See Waugh et al., 1997 for details). The tracer tends to spread out from the location where it was released.

- **Time irreversible**

  Turbulent motions are not time reversible. As time goes on, turbulent motions tend to forget their initial conditions and reach some equilibrated state. Turbulence mixes stuff up, it does not unmix it (Fig. 6). A challenge of the last part of this course will be to explain how irreversibility can arise in fluids that are governed by classical mechanics, *i.e.* Newton's dynamics, which is time reversible.

The classification of properties that a flow must display to be considered turbulent is a subject of continuous debate in the scientific community. Many authors make narrower definitions of turbulence, limiting the scope to:

- flows exhibiting explosive three dimensional vortex stretching,
- flows obeying Kolmogorov's cascade law (to be introduced next class),
- flows with a finite cascade of energy toward smaller scales.

These definitions are arbitrarily exclusive, since there are many geophysical flows which share the fundamental properties of broadband spectrum, advective nonlinearity, unpredictability, and time irreversibility, yet, due to the effects of rotation and stratification, are not fully three dimensional, do not satisfy Kolmogorov's law, and have no energy transfer toward smaller scales.

Governing equations

We will describe turbulence using the Boussinesq equations. [The Boussinesq equations follow from the full Navier-Stokes equations if one neglects all density fluctuations except those due to heat, salt, and humidity]. The ocean and the atmosphere have a more complex thermodynamics than these equations do, but this is largely extraneous to the fundamental behaviors of turbulence and thus will be ignored. The three dimensional Boussinesq system is,

$$\frac{\partial \boldsymbol{u}}{\partial t} + \underbrace{(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}}_{inertia} = -\frac{1}{\rho_0}\nabla p + \underbrace{\nu\nabla^2\boldsymbol{u}}_{friction} + \left[\underbrace{b\hat{\boldsymbol{z}}}_{buoyancy} - \underbrace{f\hat{\boldsymbol{z}}\times\boldsymbol{u}}_{Coriolis}\right], \tag{4}$$

$$\left[\frac{\partial b}{\partial t} + \underbrace{(\boldsymbol{u}\cdot\nabla)b}_{advection} = \underbrace{\kappa\nabla^2 b}_{diffusion}\right], \tag{5}$$

$$\left[\frac{\partial c}{\partial t} + \underbrace{(\boldsymbol{u}\cdot\nabla)c}_{advection} = \underbrace{\kappa\nabla^2 c}_{diffusion}\right], \tag{6}$$

$$\nabla\cdot\boldsymbol{u} = 0, \tag{7}$$

where p is the pressure, f is the Coriolis frequency associated with planetary rotation, and the vertical versor $\hat{\boldsymbol{z}}$ is assumed to be parallel to both gravity and the axis of rotation. The buoyancy b is defined in terms of density as $\rho = \rho_0(1 - g^{-1}b)$. Notice that b has the dimension of an acceleration. These equations must be complemented by forcing, boundary and initial conditions to obtain a well posed problem. We can also consider the evolution of a passive tracer c which satisfies the same equation as b, but has no influence on the evolution of $\boldsymbol{u}$.

These equations have conservative integral invariants for energy, and all powers and other functionals of buoyancy, in the absence of friction and diffusion. For non-conservative dynamics, the energy and scalar variance satisfy the equations,

$$\frac{\partial E}{\partial t} = -\epsilon, \qquad \frac{\partial B}{\partial t} = -\epsilon_B, \tag{8}$$

where,

$$[E, B, \epsilon, \epsilon_B] = \int\int\int \mathrm{d}\boldsymbol{x}\left[\frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u} - bz, b^2, \nu\,\nabla\boldsymbol{u} : \nabla\boldsymbol{u} + \kappa z\nabla^2 b, \kappa\,\nabla b\cdot\nabla b\right], \tag{9}$$

In deriving (8), it is assumed that there are no boundary fluxes of energy or scalar variance. These integrals measures of the flow can only decrease with time through the action of molecular viscosity and diffusivity. The only exception is the compressive work, $\kappa z \nabla^2 b$, which can act as a source of mechanical energy. We will return to this issue in the chapter on convection.

Every problem we will consider lies within the set of solutions of the PDE system in (4) through (7). No general solution is known, nor is any in prospect, because we do not know a mathematical methodology that seems powerful enough. However computers are giving us access to progressively better particular solutions, *i.e.* with progressively larger nonlinearities.

The brackets in (4) through (5) contain the effects of buoyancy and rotation, and these terms are ignored in the classical literature on turbulence, which deals with uniform density fluids in inertial reference frames. In these simpler circumstances, the Boussinesq system is called the incompressible Navier-Stokes equations, or with the further elimination of the frictional term, the incompressible Euler equations.

Because of the lack of general solutions to the Boussinesq equations, it is useful to identify which terms might be neglected in specific situations in order to simplify the problem and make analytical progress. The relative size of the various terms that appear in (4) through (7) can be estimated in terms of nondimensional numbers. The ones that will be most useful in this class are as follows.

- **Inertia and friction**

  The **Reynolds number** is defined as

  $$Re \equiv \frac{UL}{\nu} \tag{10}$$

  Here U and L are characteristic velocity and length scales of the flow and ν is the kinematic viscosity of the fluid. The Reynolds number measures the ratio of inertia and friction,

  $$\frac{|(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}|}{|\nu\nabla^2\boldsymbol{u}|} \approx \frac{U/L\;U}{\nu U/L^2} = Re. \tag{11}$$

  Equivalently, the Reynolds number is the ratio of the characteristic scale of the flow L and the scale at which momentum is dissipated $l = \nu/U$. In turbulent flows $Re \gg 1$, advective dominance $\Rightarrow$ nonlinear dynamics $\Rightarrow$ chaotic evolution and broadband spectrum.

  The focus of this course is on turbulence in the Earth's ocean and atmosphere. Typical values for ν near the Earth's surface are 1.5×10^{-5} m^2 s^{-1} for air and 1.0×10^{-6} m^2 s^{-1} for water. These values are small enough, given typical velocities U, that $Re \gg 1$ on all spatial scales L from the finescale of about 1 m to the planetary scale of about 10^4 km. For example, $U = 1$ m s^{-1} and $L = 10^3$ m give $Re = 10^9 - 10^{10}$ respectively for the atmosphere and the ocean.

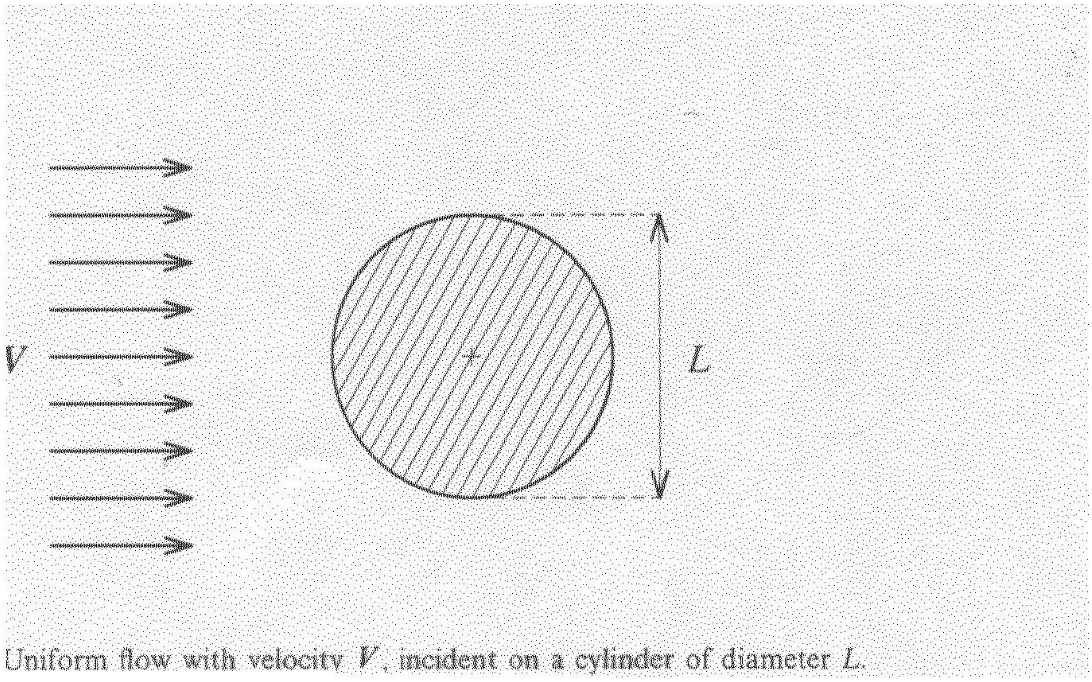


Figure 7: Uniform flow with velocity U, incident on a cylinder of diameter L.

For $Re \gg 1$, the frictional term is small, at least at the scale L. Paradoxically, however, the dissipation terms in (8) control the energetics of the system. Thus, there must be a profound difference in solutions between the asymptotic tendency as $Re \to \infty$, and the Euler limit, $Re = \infty$ or $\nu = 0$. The difference is that as long as $Re \leq \infty$, there are small scales at which friction becomes important and Re is small.

It is instructive to check how Re controls the behavior of solutions in a real flow. Let us consider a fluid of uniform density in an inertial reference system, *i.e.* let us neglect rotation and variations in buoyancy in (4) and (5) ($b = f = 0$). A classical example is a uniform flow incident on a cylinder (figure 7). Figures 8 through 10 show how the flow past the cylinder changes for different Reynolds numbers. flow that develops behind the obstacle (run dns-midi.mpg).

- **Advection and diffusion**

$$Pe \equiv \frac{UL}{\kappa} \tag{12}$$

The **Peclet number** is the direct analog of Re for a conserved tracer c with a diffusivity κ and measures the relative importance of advection and diffusion. At large Pe, the tracer evolution is dominated by advection. Once more, the limit $Pe \to \infty$ is very different from $Pe = \infty$, because dissipation, no matter

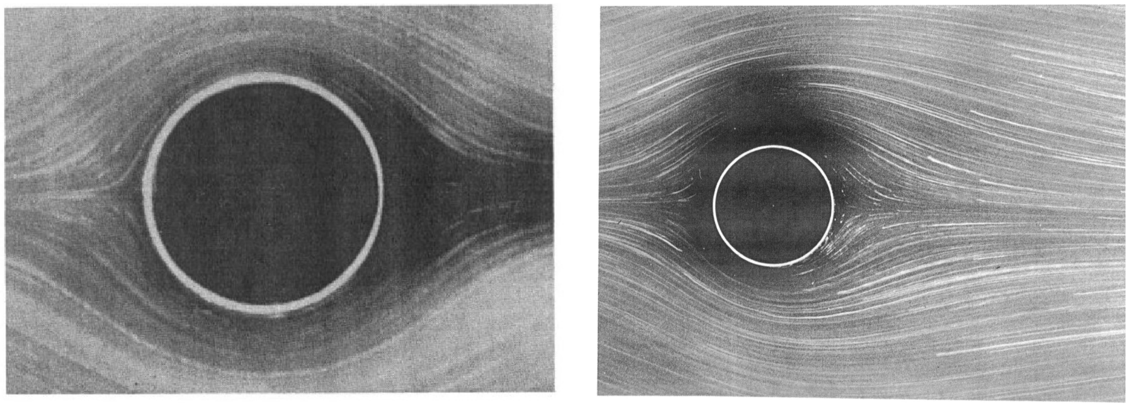


Figure 8: Flow past a cylinder at $R = 0.16$ and $R = 1.54$ (Van Dyke 1982).

how small, eventually is responsible for removing structure from the tracer field (Fig. 11).

- **Friction and diffusion**

$$Pr \equiv \frac{\nu}{\kappa} \tag{13}$$

The frictional length scale is $l_\nu = \nu/U$ and the diffusive length scale $l_\kappa = \kappa/U$. The **Prandtl number** is defined as the ratio of these two length scales, $Pr \equiv l_\nu/l_\kappa$. The Prandtl number is a property of the fluid, not of the particular flow. Hence there is a restriction on the transfer of information from experiments with one fluid to those with another. For $Pr > 1$ the scales at which friction becomes important are larger than those for diffusion and, at some small scale, we expect to find smooth velocity fields together with convoluted tracer fields. For $Pr < 1$ we expect the opposite. The Prandtl numbers for air and water are 0.7 and 12.2 respectively. The paper by Paparella and Young (Journal of Fluid Mechanics, 2002) shows examples of flows with low and high Pr.

- **Inertia and Coriolis**

$$Ro \equiv \frac{U}{fL} \tag{14}$$

The **Rossby number** Ro measures the relative importance of the real inertial forces and the fictitious Coriolis force, that appear because of the rotating reference system. Thus Ro measures the importance of rotation in the problem at hand. $Ro \gg 1$ characterizes essentially non-rotating turbulence, while $Ro \leq 1$ flows are strongly affected by rotation (Fig. 12).

- **Buoyancy and diffusion**

$$Ra \equiv \frac{\Delta b L^3}{\kappa \nu} \tag{15}$$

In convective problems, motions are generated by imposing an unstable density stratification on the fluid ($\partial b/\partial z < 0$). In these problems, it is useful to

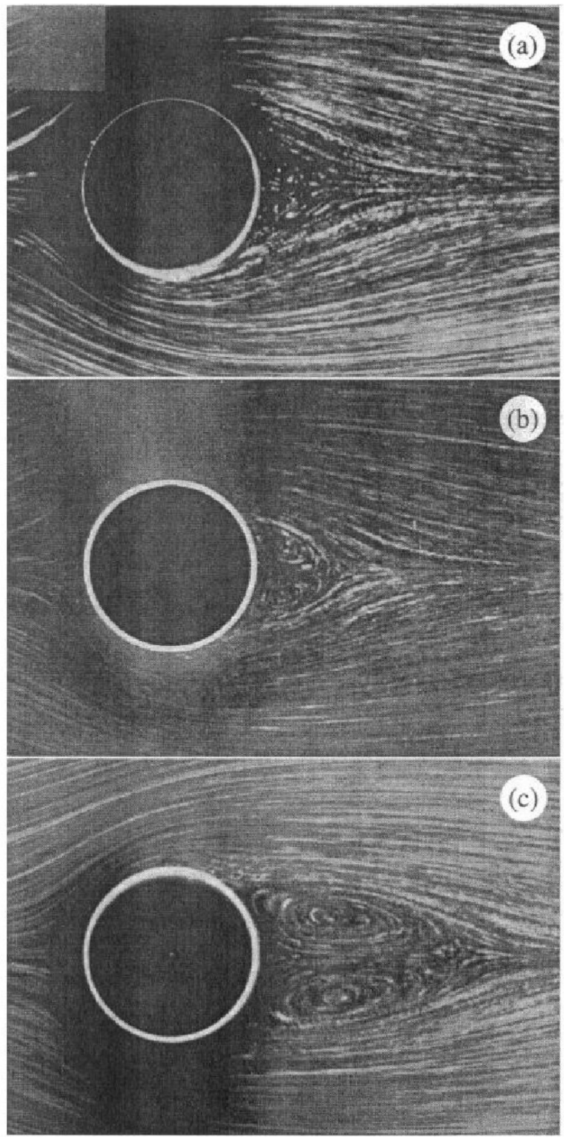


Figure 9: Flow past a cylinder at $R = 9.6$, $R = 13.1$, and $R = 26$ (Van Dyke 1982)

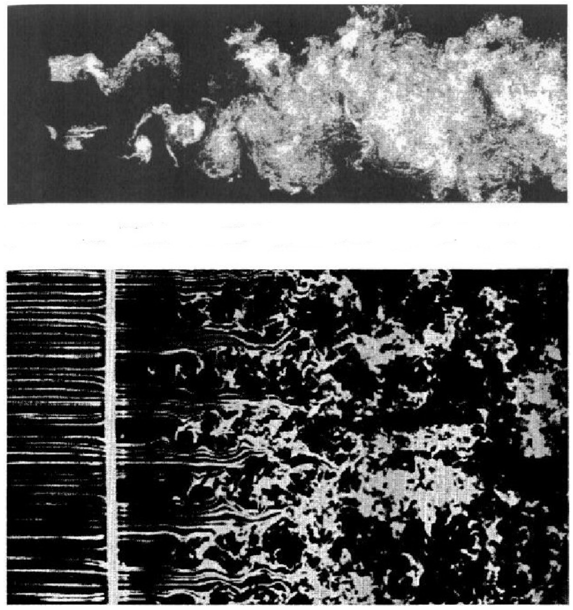


Figure 10: Wake behind two cylinders at $R = 1800$, and homogeneous turbulence behind a grid at $R = 1500$ (Van Dyke 1982).

characterize turbulence in terms of the **Rayleigh number**, *i.e.* the ratio of the diffusive timescale $t_\kappa = L^2/\kappa$ and the buoyancy timescale $t_b = (L/\Delta b)^{1/2}$. The buoyancy scale Δb is the buoyancy difference maintained across the layer depth L through external forcing. If the forcing is imposed by maintaining a temperature difference ΔT, then one has $\Delta b = g\alpha\Delta T$, where α is the coefficient of thermal expansion of the fluid, and g the acceleration of gravity. Convection starts if $t_\kappa \gg t_b$, *i.e.* if $RaPr \gg 1$, when diffusion is too slow to change substantially the buoyancy of water/air parcels as they rise (Fig. 13).

- **Buoyancy and inertia**

$$Ri \equiv \frac{\partial b/\partial z}{|\partial \boldsymbol{u}/\partial z|^2} \tag{16}$$

In the presence of stable buoyancy stratification, vertical motions tend to be suppressed, but turbulence can still emerge, if there is enough energy in the horizontal velocity field. A useful parameter to characterize the flow in these problems is the ratio of the buoyancy timescale $t_b = (L/\Delta b)^{1/2} = 1/(\partial b/\partial z)^{1/2}$ and the inertial timescale due to horizontal shears in the flow $t_i = L/U = 1/(\partial \boldsymbol{u}/\partial z)$. This ratio is called the gradient Richardson number Ri. If $Ri \ll 1$, buoyancy can be neglected in the momentum equations, and it becomes a passive scalar with no feedbacks on the dynamics (Fig. 14).

A final remark about the only term that never appeared explicitly in the nondimensional numbers presented: the pressure force. Pressure can be formally eliminated

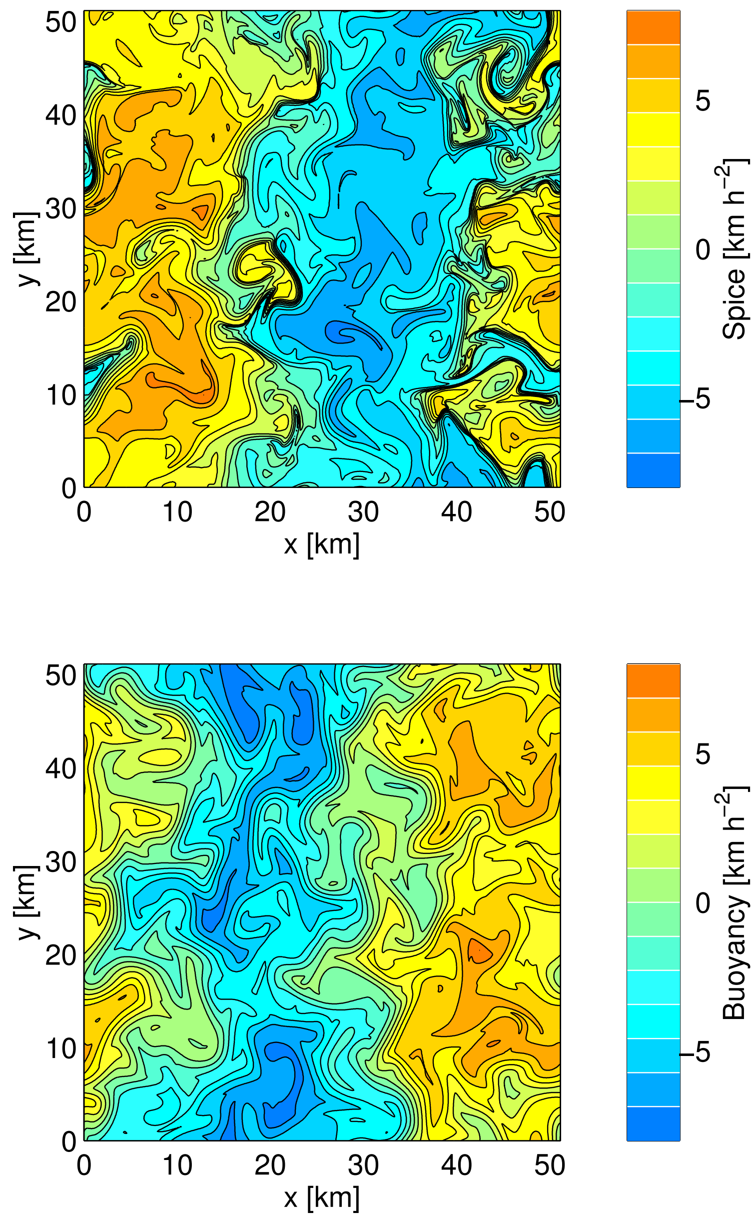


Figure 11: Contours of two tracers advected by a two dimensional flow field. The two tracers have $Pe = 100$ (upper panel) and $Pe = 10$ (lower panel).

Figure 12: Rotational effects produce the well-known jets in the atmosphere of Jupiter. The Ro can be estimated to be of order 0.1 using $U = 300$ m/s, $f = 2.5 \cdot 10^{-4}$ s^{-1}, and $L = 10,000 km$).

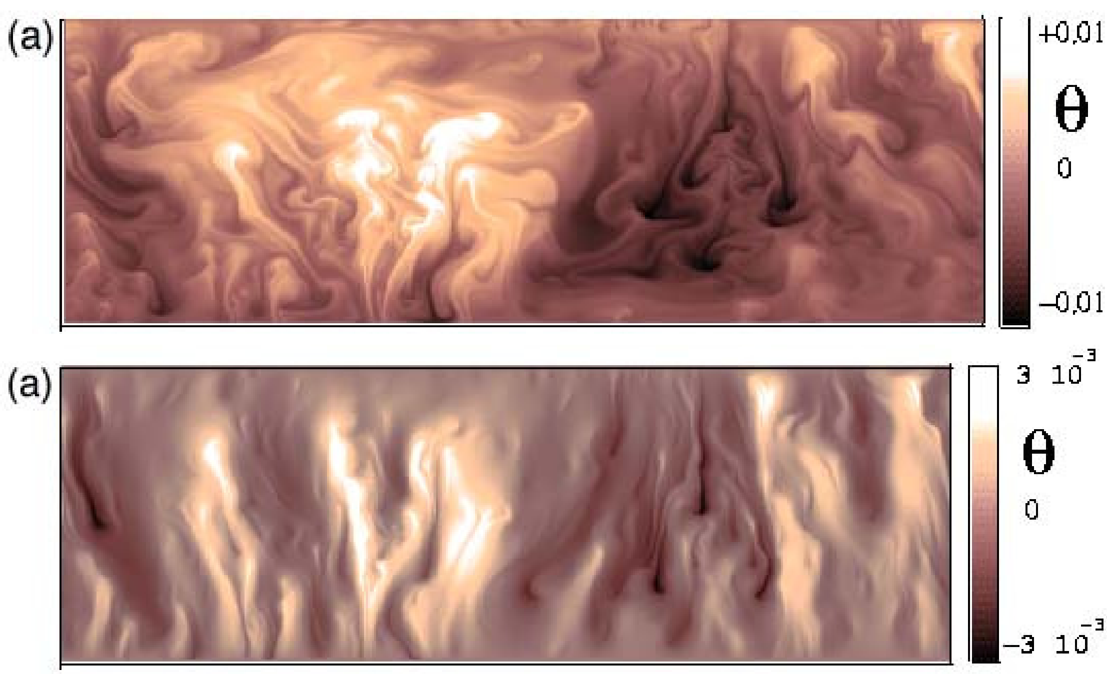


Figure 13: Convection in air at a $Ra = 10^{12}$ and $Ra = 10^{14}$ (Alain Vincent and David A. Yuen, Phys. Review E, 2000).

Figure 14: Kelvin Helmholtz instability in the atmosphere. Picture taken near Washington, D.C.. Darker areas are clouds.

from the equations. This is a consequence of the Boussinesq approximation. We simply need to take the divergence of the momentum equation in (4) and note that $\nabla \cdot \boldsymbol{u}_t = 0$ because of incompressibility. This yields the relation,

$$\nabla^2 p = \rho_0 \nabla \cdot \Big[-(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nu \nabla^2 \boldsymbol{u} + b\hat{\boldsymbol{z}} - f\hat{\boldsymbol{z}} \times \boldsymbol{u} \Big] . \tag{17}$$

Since there are no time derivatives in (17), pressure is a purely diagnostic field, which is wholly slaved to $\boldsymbol{u}$. It can be calculated from (17) and then substituted for the pressure gradient force in the momentum equations. Its role is to maintain incompressibility under the action of all other forces. It would be redundant to introduce nondimensional parameters involving pressure, because any such parameters can be expressed as combinations of the parameters already discussed.

A statistical description of turbulence

The evolution of turbulent flows is very complex. Turbulent flows appear highly disorganized with structure at all scales. Signals appear unpredictable in their detailed behavior. However some statistical properties of the flow can be quite reproducible. This suggests that it can be useful to seek a statistical description of turbulent flows. A statistical measure is an average of some kind: over the symmetry coordinates, if any are available (*e.g*, a time average for stationary flows); over multiple realizations (*e.g*, an ensemble); or over the phase space of solutions if the dynamics are homogeneous.

A similar behavior is observed in simple deterministic maps. Frisch, is chapter 3 of his book *Turbulence*, provides examples of deterministic maps that are chaotic and not predictable in their detailed properties, but whose statistical properties are reproducible, just like for turbulent flows.

Thus it seems quite appropriate to introduce a probabilistic description of turbulence. However we know that the basic Boussinesq equations are deterministic. How can chance and chaos arise in a purely deterministic context? A nice discussion of this issue can be found, once more, in chapter 3 of Frisch's book.

Ilya Prigogine has in recent years brought about a radical change of perspective. The statistical description of turbulence is not merely a convenience to describe the excessive amount of information contained in the fluid. Turbulence is intrinsically stochastic. The argument goes that single trajectories of fluid parcels in phase space are deterministic, but a fluid composed by a large ensemble of parcels is not. All parcels interact in a such a way that information is continuously spread, and the ensemble evolves toward a collective state that can be defined only statistically (like thermodynamics). This description suggests that irreversibility appears in nature as a result of the statistical behavior of parcel interactions.

Even though Frisch (and most of the scientific community) and Prigogine disagree in their explanation of why a statistical description of turbulence is appropriate, it is clear that they both agree that turbulence and statistics go hand in hand. Thus we will spend the rest of this lecture to illustrate how statistical tools can be applied to describe turbulent flows.

The poor man's Navier-Stokes map

Following Frisch we introduce a discrete map that mimics some of the properties of the Navier-Stokes equations: the poor man's Navier-Stokes equation. The analogy is most apparent if we discretize the Navier-Stokes equation in time,

$$\underbrace{\frac{v_{n+1} - v_n}{\tau}}_{\boldsymbol{v}_t} = - \underbrace{\frac{v_n^2}{\lambda}}_{-\boldsymbol{u}\cdot\nabla\boldsymbol{u}+\nabla p} - \underbrace{\nu\frac{v_n}{\lambda^2}}_{+\nu\nabla^2\boldsymbol{u}} +f. \tag{18}$$

With appropriate choice of time step τ, length scale λ, friction ν, and forcing f, we can reduce the map to,

$$v_{n+1} = -2v_n^2 + 1. \tag{19}$$

This map displays broadband spectrum in time, nonlinearity, unpredictable behavior, and time irreversibility (Fig. 15). However it is derived assuming a specific lengthscale λ and it cannot display a broadband spatial spectrum. Thus the poor man's Navier-Stokes equation cannot display turbulent behavior. However it is a useful tool to study how a deterministic system can produce chaos and unpredictable behavior. Properties of this map are:

- the signal is very disorganized
- trajectories are unpredictable
- histogram of positions is quite reproducible

The poor man's linear Navier-Stokes equation

We also consider a linear version of the poor man's Navier-Stokes equation,

$$v_{n+1} = -\frac{1}{2}v_n + 1. \tag{20}$$

Properties of this map are:

- the signal is very organized
- trajectories are predictable
- histogram of positions and trajectories are complementary descriptions

In this map all trajectories collapse to a fixed point $v = 2/3$. The histogram collapses to a spike centered at $v = 2/3$.

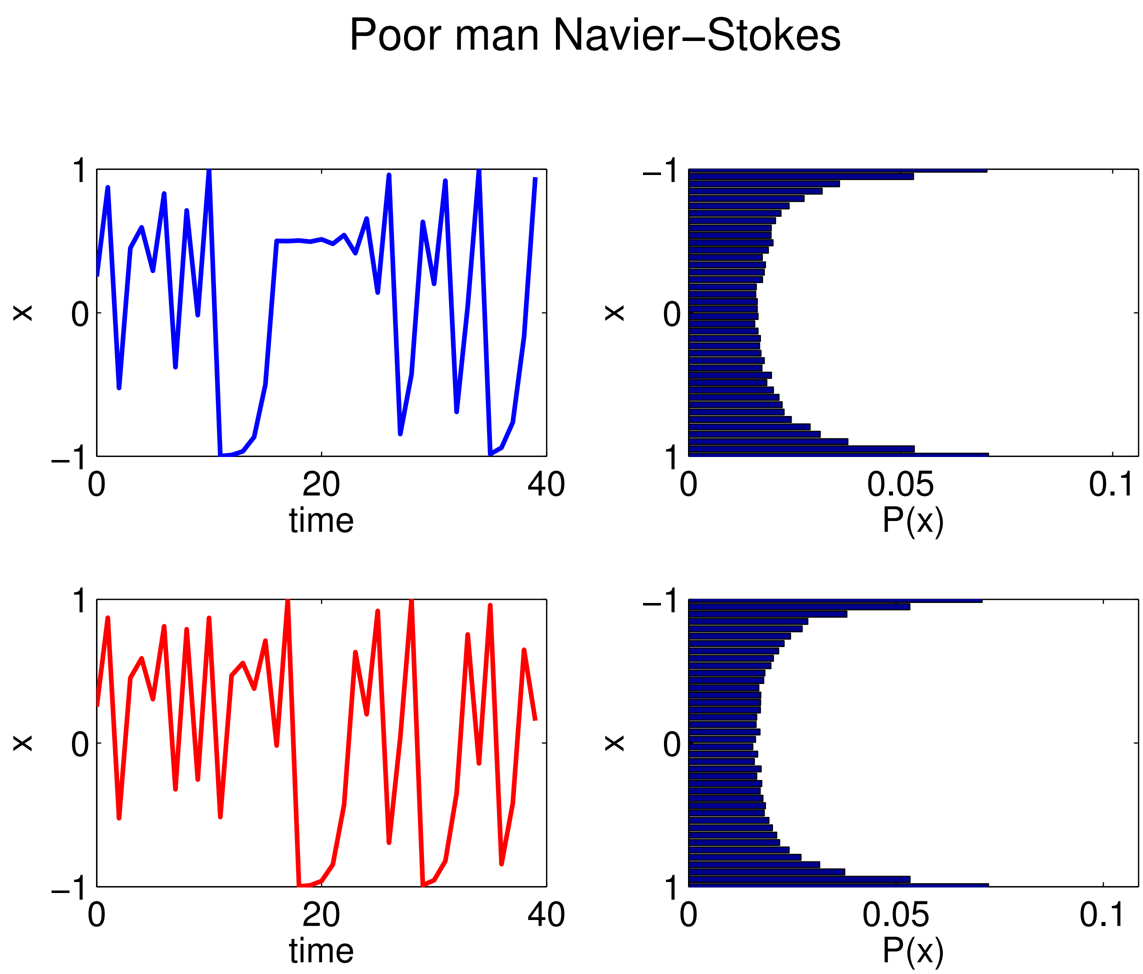


Figure 15: Trajectories and histograms for the poor man's Navier-Stokes equation .

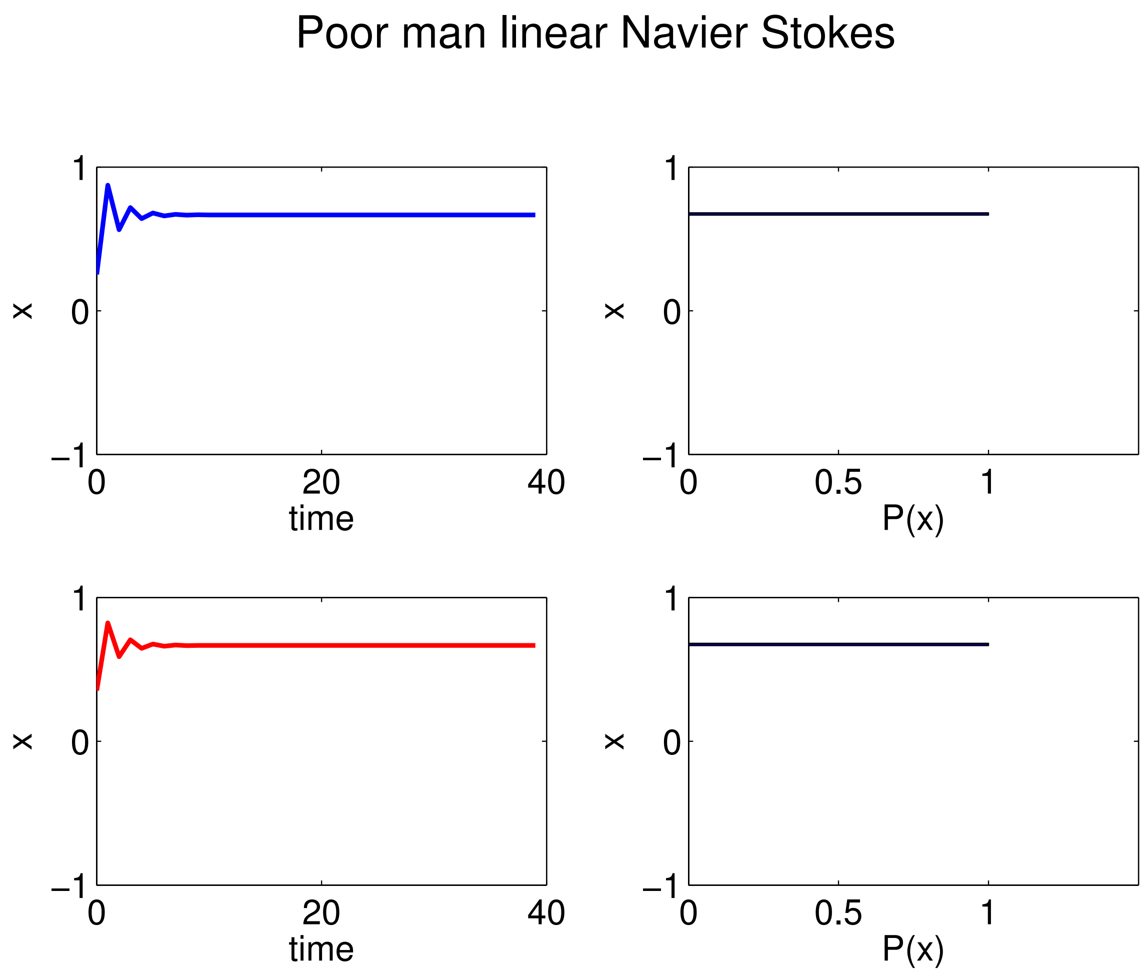


Figure 16: Trajectories and histograms for the poor man's linear Navier-Stokes equation.

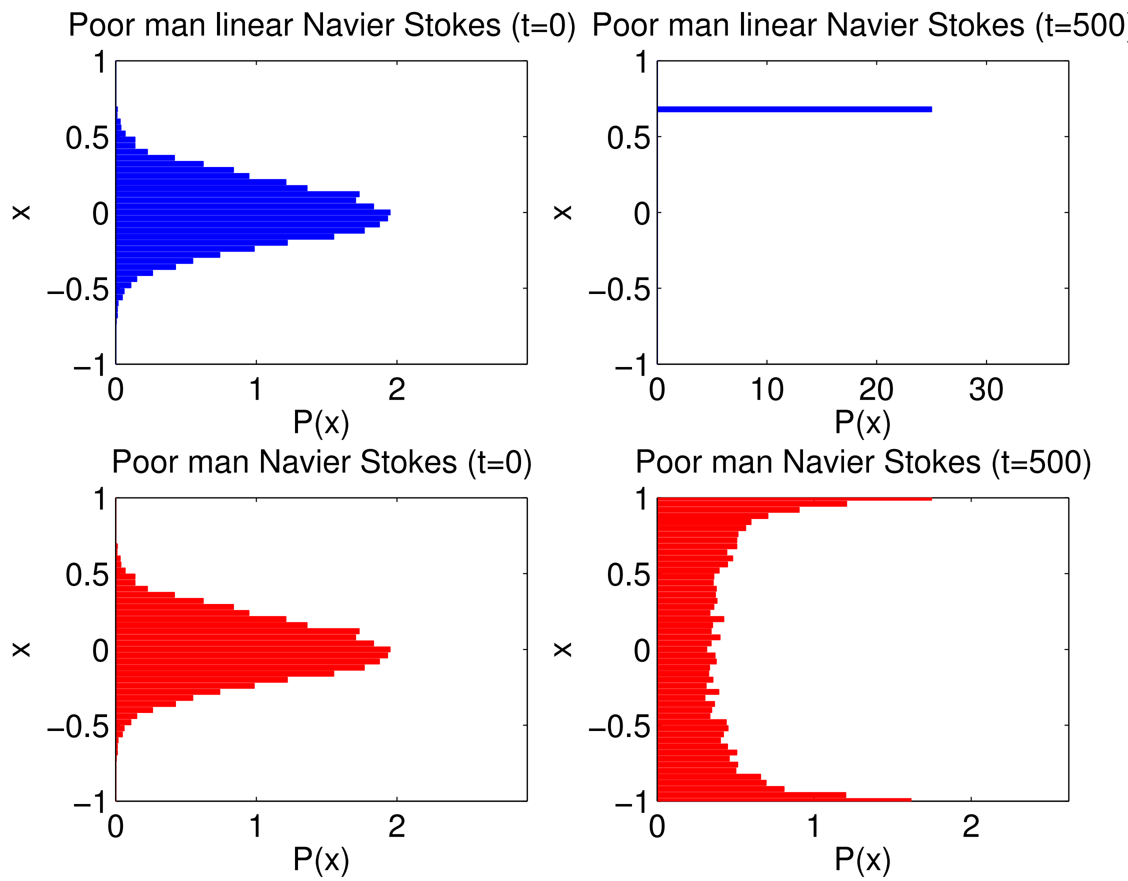


Figure 17: The evolution of the histograms for the histograms for the poor man's linear Navier-Stokes equation and the poor man's Navier-Stokes equation .

Trajectory and histograms of maps

The two previous examples suggest that the histograms and trajectories contain the same information for the linear map. However in the nonlinear map the histograms show a predictability that does not emerge in individual trajectories. The histogram always converge toward the same distribution, regardless of initial conditions (Fig. 17). Why does the histogram of the poor man's Navier-Stokes equation converge to a limit solution? Why a deterministic system such as a map has a regular statistical behavior? We do not have complete answers to these equations, but progress is being made. The current understanding is that as time progresses the trajectory explores the whole phase space and gathers information about all other trajectories. Thus information is continuously spread, and the ensemble evolves toward a collective state that can be defined only statistically.

A probabilistic description of maps

In this section we derive a statistical description of the maps we considered. The goal is to understand the difference between the trajectory-based description and the histogram-based description. First we need to determine the histogram $P(v)$, also known as the probability distribution function, so that we can observe the recurrence relation $P_{n+1}(v) = UP_n(v)$. The distribution function $P_{n+1}(v)$ after $n+1$ iterations is obtained by the action of the operator U on $P_n(v)$, which is the distribution function after n iteration of the map. The operator U acts on functions and it is known as the Perron-Frobenius operator.

We can derive the Perron-Frobenius operator for the poor man's linear Navier-Stokes map. Simple algebra gives,

$$v_{n+1} = 1 - \frac{1}{2}v_n \quad \Longleftrightarrow \quad P_{n+1}(v) = 2P_n(2-2v)\,. \tag{21}$$

Equilibrium solutions correspond to probability functions that do not change under the action of the map, i.e. $P_{n+1}(v) = P_n(v)$. For the poor man's linear Navier-Stokes map the only stationary solution is a Dirac delta function centered at $2/3$. This is equivalent to the result that all trajectories collapse to $v = 2/3$. No new information is gained by looking at the probability distribution.

In order to derive the Perron-Frobenius operator for poor man's Navier-Stokes map, it is useful to simplify the map through a change of variables,

$$v_n = \sin\left(\pi x_n - \frac{\pi}{2}\right), \qquad 0 \le x_n \le 1, \tag{22}$$

and similarly Navier-Stokes map to a simpler map. Let us make the following change of variable,

$$v_{n+1} = \sin\left(\pi x_{n+1} - \frac{\pi}{2}\right), \qquad 0 \le x_n \le 1, \tag{23}$$

It is left as an exercise to prove that the map for x_n is,

$$x_{n+1} = \begin{cases} 2x_n, & 0 \le x_n \le 1/2 \\ 2 - 2x_n, & 1/2 \le x_n \le 1 \end{cases} \tag{24}$$

This is known as the *tent map*, because of the shape of its graph. It is now easy to understand why this map displays sensitive dependence on initial conditions. See Frisch's book for details.

We can write the Perron-Frobenius operator for the tent map,

$$P_{n+1}(v) = \frac{1}{2}\left[P_n\left(\frac{v}{2}\right) + P_n\left(1 - \frac{v}{2}\right)\right]. \tag{25}$$

As a consequence of the form of the Perron-Frobenius operator, if P_n is constant equal to P_0, then P_{n+1} is also equal to P_0. The uniform distribution $P = P_0$ is the equilibrium distribution. The uniform distribution is indeed the final state that one obtains by running numerical integrations of the tent map. Thus the statistical description predicts a result that cannot be derived from simple inspection of the deterministic equation.

How do we know that the equilibrium distribution is obtained for any set of initial conditions? In order to solve arbitrary initial value problems, we need the full set of eigenfunctions and eigenvalues of the Perron-Frobenius operator. These eigenfunctions can be used to represent any arbitrary initial condition. The eigenfunctions for the tent map belong to a family of polynomials called the *Bernoulli polynomials*. The eigenfunctions are found by solving the problem,

$$P(v) = \lambda \, UP(v). \tag{26}$$

In the example of the tent map, we find that the eigenfunctions have $\lambda \leq 1$. For example an eigenfunction is given by,

$$P(v) = v^2 - 2v + \frac{2}{3}. \tag{27}$$

This eigenfunction has an eigenvalues $\lambda = 1/4$. The uniform distribution $P = 1$ is the only eigenfunction with eigenvalue $\lambda = 1$. Thus all eigenfunctions other than the uniform distribution decay in time. And the uniform distribution emerges as the asymptotic state.

It is left as an exercise to relate the probability distribution of the tent map to that of the poor man's Navier-Stokes map, and show that it correctly predicts what we found by numerical integrations of the map.

Shortcoming of the poor man's Navier-Stokes analogy

The poor man's Navier-Stokes map is a useful tool to illustrate some important characteristics of turbulent flows. However this tool is pathological in at least two ways,

- The poor man's Navier-Stokes map explores the full available space $[-1, 1]$. Typically turbulent systems are dissipative and collapse on an attractor with fractal structure (at least for finite-dimensional systems).
- Natural systems tend to have more than one attractor. Thus the equilibrium statistical properties are not fully predictable.

Furthermore there are technical issues that have not been solved for the Navier-Stokes equations. We do not know if solutions exist for all times for arbitrary initial condition. We do not know how to write the equivalent of the Perron-Frobenius operator.

The Closure Problem

Although it is impossible to predict the detailed motion of each eddy in a turbulent flow, the mean state may not be changing. For example, consider the weather system, in which the storms, anti-cyclones, hurricanes, fronts etc. constitute the eddies. Although we cannot predict these very well, we certainly have some skill at predicting their mean state, the climate. For example, we know that next summer will be warmer than next winter, and that in California summer will be drier than winter. We know that next year it will be colder in Canada than in Mexico, although there might be an occasional day when this is not so. We would obviously like to be able to predict the mean climate without necessarily trying to predict or even simulate all the eddies. We might like to know what the climate will be like one hundred years from now, without trying to know what the weather will be like on February 9, 2056, plainly an impossible task. Even though we know what equations determine the system, this task proves to be very difficult because the equations are nonlinear. This is the same problem we discussed at the beginning of the lecture. We seek a statistical description of the turbulent flow, because a detailed description is beyond our grasp. The simplest statistical quantity we might try to predict is the mean flow. However, because of the nonlinearities in the equations, we come up against the closure problem.

The program is to first decompose the velocity field into mean and fluctuating components,

$$\mathbf{u} = \bar{\mathbf{u}} + \mathbf{u}'. \tag{28}$$

Here $\boldsymbol{u}$ is the mean velocity field, and $\mathbf{u}'$ is the deviation from that mean. The mean may be a time average, in which case $\bar{\mathbf{u}}$ is a function only of space and not time. It might be a time mean over a finite period (e.g a season if we are dealing with the weather). Most generally it is an ensemble mean. Note that the average of the deviation is, by definition, zero; that is $\overline{\mathbf{u}'} = 0$. We then substitute into the momentum equation and try to obtain a closed equation for $\bar{\mathbf{u}}$. To visualize the problem most simply, we carry out this program for a model nonlinear system, proposed by Geoff Vallis in his GFD book, which obeys,

$$\frac{du}{dt} + uu + \nu u = 0. \tag{29}$$

The average of this equation is,

$$\frac{d\bar{u}}{dt} + \overline{uu} + \nu\bar{u} = 0. \tag{30}$$

The value of the term $\overline{uu}$ is not deducible simply by knowing $\bar{u}$, since it involves correlations between eddy quantities $\overline{u'u'}$. That is, $\overline{uu} = \bar{u}\bar{u} + \overline{u'u'} \neq \bar{u}\bar{u}$. We can go to next order to try to obtain a value for $\overline{uu}$. First multiply (29) by u to obtain an equation for u^2 , and then average it to yield,

$$\frac{1}{2}\frac{d\overline{uu}}{dt} + \overline{uuu} + \nu\overline{uu} = 0. \tag{31}$$

This equation contains the undetermined cubic term $\overline{uuu}$. An equation determining this would contain a quartic term, and so on in an unclosed hierarchy. Most current methods of closing the hierarchy make assumptions about the relationship of (n+1)-th order terms to n-th order terms, for example by supposing that,

$$\overline{uuuu} = \overline{uu}\ \overline{uu} - \alpha\overline{uuu}, \tag{32}$$

where α is some parameter. Such assumptions require additional, and sometimes dubious, reasoning. Nobody has been able to close the system without introducing physical assumptions not directly deducible from the equations of motion.

The Reynolds equations

Let us repeat the averaging procedure for the full Boussinesq equations. We start with the momentum equations,

$$\frac{\partial\bar{\boldsymbol{u}}}{\partial t} + (\bar{\boldsymbol{u}}\cdot\nabla)\bar{\boldsymbol{u}} + f\hat{\boldsymbol{z}}\times\bar{\boldsymbol{u}} = \bar{b}\hat{\boldsymbol{z}} - \frac{1}{\rho_0}\nabla\bar{p} + \nu\nabla^2\bar{\boldsymbol{u}} - \overline{(\boldsymbol{u}'\cdot\nabla)\boldsymbol{u}'}, \tag{33}$$

The extra term on the right hand side represent the effect of eddy motions on the mean flow. If the average operator is a time average over some time T, then eddy motions are those motions with time scales shorter than T. If the average operator is a spatial average over some scale L, then eddy motions are those motions with spatial scales shorter than L. If the average operator is an ensemble mean, then the eddy motions are those motions that change in every realizations, regardless of their scale, i.e. they represent the unpredictable or turbulent part of the flow.

Using the continuity equation,

$$\nabla\cdot\boldsymbol{u} = 0 \qquad \Rightarrow \qquad \nabla\cdot\bar{\boldsymbol{u}} = 0 \quad \text{and} \quad \nabla\cdot\boldsymbol{u}' = 0, \tag{34}$$

we can rewrite the averaged momentum equation as,

$$\frac{\partial\bar{\boldsymbol{u}}}{\partial t} + (\bar{\boldsymbol{u}}\cdot\nabla)\bar{\boldsymbol{u}} + f\hat{\boldsymbol{z}}\times\bar{\boldsymbol{u}} = \bar{b}\hat{\boldsymbol{z}} - \frac{1}{\rho_0}\nabla\cdot\left[\bar{p}\mathbf{I} - \rho_0\nu\nabla\bar{\boldsymbol{u}} + \rho_0\overline{\boldsymbol{u}'\boldsymbol{u}'}\right]. \tag{35}$$

$\mathbf{I}$ is th unit matrix. These are the so-called **Reynolds momentum equation** and the eddy flux $\rho_0\overline{\boldsymbol{u}'\boldsymbol{u}'}$ represent the **Reynolds stress tensor** due to fluctuations in velocity filed.

We can similarly decompose the buoyancy equation into a mean and a fluctuating component, $b = \bar{b} + b'$, and write an equation for the mean component by substituting back into the buoyancy equation,

$$\frac{\partial \bar{b}}{\partial t} + (\bar{\boldsymbol{u}} \cdot \nabla)\bar{b} = -\nabla \cdot \left[-\kappa \nabla \bar{b} + \overline{\boldsymbol{u}'b'}\right]. \tag{36}$$

The eddy term $\overline{\boldsymbol{u}'b'}$ represent the **Reynolds eddy flux** of buoyancy.

The problem of turbulence is often that of finding a representation of such Reynolds stress and flux terms in terms of mean flow quantities. However, it is not at all clear that a general solution (or parameterization) exists, short of computing the terms explicitly.

Eddy viscosity and eddy diffusivity

The simplest closure for the Reynolds stress terms is one which relates $\overline{\boldsymbol{u}'\boldsymbol{u}'}$ to the mean flow, by assuming a relation of the form,

$$\overline{\boldsymbol{u}'\boldsymbol{u}'} = -\nu_T \nabla \bar{\boldsymbol{u}}, \tag{37}$$

where ν_T is the **eddy viscosity**. With such a closure the Reynolds stress term takes the same form as the mean viscosity term, but with a different viscosity. In essence, this closures states that turbulent eddies are similar to molecular motions that constantly act to redistribute and homogenize momentum. Similarly, for the tracer flux term we can define an **eddy diffusivity**

$$\overline{\boldsymbol{u}'b'} = -\kappa_T \nabla \bar{b}. \tag{38}$$

This eddy viscosity/diffusivity closure is the most commonly used in modeling and interpretation of geophysical observations. At the crudest level κ_T and ν_T are assumed to be constants; in more sophisticated models they are functions of the large scale flow. However, an eddy viscosity/diffusivity closure is rarely appropriate.

(Further reading: Chapter 1 of McComb)

Mixing length model

Prandtl provided a heuristic rationalization for the eddy viscosity/diffusivity closure. Here we follow the derivation for the eddy viscosity ν_T, but the derivation can be repeated for any passive scalar as well.

Consider a parcel in the shear flow introduced above, $(\bar{u}(y), 0, 0)$. Assume that the parcel was initially at at some position y. If the parcel moves due to turbulent motion, up to a position $y + \delta y$, and it conserves momentum, then it has a momentum deficit compared to the parcels around it of,

$$u' = [\bar{u}(y) - \bar{u}(y + \delta y)] + \delta u \approx -\delta y \frac{\partial \bar{u}}{\partial y} + \delta u, \tag{39}$$

$$v' = \delta v, \tag{40}$$

where δu and δv are the random velocity fluctuations that every particle experience. Notice that we also had to assume,

$$\delta y \ll \frac{\partial \bar{u}/\partial y}{\partial^2 \bar{u}/\partial y^2}, \tag{41}$$

in order to neglect higher order terms in the Taylor series expansion. If we further assume that the statistics of turbulent fluctuations are homogeneous and isotropic,

$$\overline{u'v'} = -\overline{\delta y \delta v} \frac{\partial \bar{u}}{\partial y}. \tag{42}$$

Introducing the **mixing length** ℓ - the distance at which δv and δy become uncorrelated - we can write,

$$\overline{\delta y \delta v} = -c \; \ell \; \sqrt{\overline{\delta v^2}}, \tag{43}$$

where c is a constant. We then have,

$$\overline{u'v'} = -\nu_T \frac{\partial \bar{u}}{\partial y} = -c \; \ell \; \sqrt{\overline{\delta v^2}} \frac{\partial \bar{u}}{\partial y}, \tag{44}$$

where ν_T is the eddy viscosity,

$$\nu_T = c \; \ell \; \sqrt{\overline{\delta v^2}}. \tag{45}$$

Under the isotropy assumption $\overline{\delta v^2} = \overline{\delta u^2} = \overline{\delta w^2}$, we can write this as

$$\nu_T = c_\mu \; \ell \; \sqrt{q} \tag{46}$$

where $q/2$ is the small-scale turbulent kinetic energy, TKE, and c_μ is again a constant. Eq.(46) could also be obtained on dimensional grounds, by assuming the turbulent motion is characterized by a single velocity scale $\sqrt{q}$, and a single lengthscale ℓ.

The problem of estimating ν_T is now reduced to one of estimating the TKE and the mixing length ℓ. Notice that if q and ℓ change in space, it seems that we have to abandon the assumption that the turbulence is homogeneous. But homogeneity is at the core of eddy mixing length arguments. A way out of this apparent inconsistency is to assume that turbulence is homogeneous on scales smaller than the model grid size and thus we can apply mixing length theory. However variations in turbulent levels appear on the much larger scales explicitly resolved by the model and this is why we need equations for q and ℓ or q and ϵ.

A second issue of concern is that eddy mixing length theory should not be used for non-conserved quantities. If we assume that the average is carried on distances so short that pressure effects do not change momentum much, then we can apply eddy mixing length theory to momentum. However this is often done for models whose resolution is too coarse for this to be true. The only rationale to use a large effective viscosity in these cases is numerical: without a large ν_T coarse resolution models tend to be unstable.