12.802
Small Scale Ocean Dynamics

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Kinetic energy and vorticity budgets in 3D turbulence

The mean zonal velocity is zero by definition of homogeneity and isotropy. The theory of 3D isotropic homogeneous turbulence is therefore a theory of the second order statistics of the velocity field, i.e. the kinetic energy $E \equiv 1/2\langle u_i^2 \rangle$. The angle brackets represent an ensemble average.

In order to understand the energy balance of 3D flows, it is useful to derive the vorticity equation. From the Navier-Stokes equations for a homogeneous fluid,

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{\zeta} \times \boldsymbol{u} = -\nabla \left(p + \frac{1}{2} \boldsymbol{u} \cdot \boldsymbol{u} \right) - \nu \nabla \times \boldsymbol{\zeta}, \tag{1}$$

$$\nabla \cdot \boldsymbol{u} = 0, \qquad \boldsymbol{\zeta} = \nabla \times \boldsymbol{u}, \tag{2}$$

we can derive the vorticity equation,

$$\frac{\partial \boldsymbol{\zeta}}{\partial t} + \nabla \times (\boldsymbol{\zeta} \times \boldsymbol{u}) = \nu \nabla^2 \boldsymbol{\zeta}. \tag{3}$$

The energy equation can be written as,

$$\frac{\partial E}{\partial t} + \nabla \cdot [\boldsymbol{u}(p + E)] = -\nu \boldsymbol{u} \cdot \nabla \times \boldsymbol{\zeta}, \tag{4}$$

or

$$\frac{\partial E}{\partial t} + \nabla \cdot [\boldsymbol{u}(p + E) + \nu \boldsymbol{\zeta} \times \boldsymbol{u}] = -2\nu Z, \tag{5}$$

with $E \equiv 1/2(\boldsymbol{u} \cdot \boldsymbol{u})$ being the energy and $Z \equiv 1/2(\boldsymbol{\zeta} \cdot \boldsymbol{\zeta})$ being the enstrophy. Turbulence in the presence of boundaries can flux energy through the walls, but for homogeneous turbulence the flux term must on the average vanish and the dissipation is just $2\nu Z$. The cascade of energy down to dissipation scales is closely tied to the rotational nature of a turbulent flow.

Taking an ensemble average, denoted with angle brackets, the energy budget for an isotropic homogeneous flow becomes,

$$\frac{\partial \langle E \rangle}{\partial t} = -\epsilon, \tag{6}$$

where $\epsilon \equiv 2\nu\langle Z \rangle$. This equation state that the rate of change of turbulent kinetic energy E is balanced by viscous dissipation ϵ. Such a balance cannot be sustained for long times - a source of kinetic energy is needed. However sources of TKE are typically not homogeneous: think of a stirrer or an oscillating boundary. We sidestep this contradiction by assuming that for large Reynolds numbers, although isotropy and homogeneity are violated by the mechanism producing the turbulence, they still hold at small scales and away from boundaries. In this classic picture, we force the energy at a certain rate, which is then balanced off by dissipation. But the energy injection does not depend upon viscosity. For the balance (6) to hold, the enstrophy must grow as the viscosity decreases. The enstrophy equation indeed has suitable source terms,

$$\frac{\partial \langle Z \rangle}{\partial t} + \nabla \cdot \langle \boldsymbol{u} Z - \nu \nabla Z \rangle = \langle \zeta_i S_{ij} \zeta_j \rangle - \langle \nu (\partial_i \zeta_j)^2 \rangle \tag{7}$$

where $S_{ij} = 1/2[\partial_i u_j + \partial_j u_i]$ is the rate of strain tensor. Therefore, if the vorticity vector is on average aligned with the directions where the strain is causing extension rather than contraction, the enstrophy will increase. In other words, we expect constant-vorticity lines to get longer on average. In a 3-D turbulent flow, the vorticity is indeed on average undergoing stretching since this term ends up balancing the sign-definite dissipation terms.

Dhar (Phyisics of Fluis, **19**, 1976) gave a simple proof of why we might expect that the average length of constant-voticity lines increases in a statistically isotropic velocity field. Consider two neighboring fluid particles with position vectors $\boldsymbol{r}_1(t)$ and $\boldsymbol{r}_2(t)$. Let $\delta\boldsymbol{r} = \boldsymbol{r}_2(t) - \boldsymbol{r}_1(t)$ be the infinitesimal displacement between the two moving particles. Then,

$$\frac{D\delta\boldsymbol{r}}{Dt} = \frac{D\boldsymbol{r}_2}{Dt} - \frac{D\boldsymbol{r}_1}{Dt}. \tag{8}$$

If $\delta\boldsymbol{r}$ is small, this gives,

$$\frac{D\delta\boldsymbol{r}}{Dt} = \boldsymbol{u}(\boldsymbol{r}_2, t) - \boldsymbol{u}(\boldsymbol{r}_1, t) = (\partial_j \boldsymbol{u})\delta r_j. \tag{9}$$

This equation is identical to the vorticity equation in the absence of dissipation,

$$\frac{D\boldsymbol{\zeta}}{Dt} = (\partial_j \boldsymbol{u})\zeta_j. \tag{10}$$

We can therefore think that the vorticity is "frozen into the fluid". The field $\boldsymbol{\zeta}$ can be stretched and bent, but never thorn apart. Its topology is preserved despite distortion. We can now proceed to analyze the evolution of $\delta \boldsymbol{r}$ and we will use the results to understand the evolution of $\boldsymbol{\zeta}$.

Let $\delta r_i(t) = \delta r_i(0) + \Delta r_i(t)$. The initial particle positions are given and non-random (i.e. statistically sharp), but the subsequent locations acquire a statistical distribution that depends on the statistics of the velocity field. the mean square separation between the fluid particles at time t is thus,

$$\langle |\delta \boldsymbol{r}|^2 \rangle = |\delta \boldsymbol{r}(0)|^2 + 2\delta \boldsymbol{r}(0) \cdot \langle \Delta \boldsymbol{r}(t) \rangle + \langle |\Delta \boldsymbol{r}(t)|^2 \rangle. \tag{11}$$

For a statistically homogeneous and isotropic flow,

$$\langle \Delta \boldsymbol{r}(t) \rangle = 0 \tag{12}$$

and therefore,

$$\langle |\delta \boldsymbol{r}(t)|^2 \rangle \geq |\delta \boldsymbol{r}(0)|^2. \tag{13}$$

The distance between fluid particles increases on average! Unfortunately, however, this proof does not, strictly speaking, apply to the case of constant-vorticity lines, because (12) holds only for fluid particles whose relative initial locations are initially uncorrelated with the flow (and therefore does not apply to pairs of fluid particles that always lie on a vorticity line). Thus (12) contributes plausibility, but not rigor, to the idea that in a turbulent flows constant vorticity lines are stretched on average.

Kinetic Energy Spectra for 3D turbulence

0.1 Definition of KE in spectral space

For a flow which is homogeneous in space (i.e. statistical properties are independent of position), a spectral description is very appropriate, allowing us to examine properties as a function of wavelength. The average total kinetic energy can be written as,

$$E = \frac{1}{2}\frac{1}{V}\iiint \langle u_i(\mathbf{x}) u_i(\mathbf{x}) \rangle \, \mathrm{d}\mathbf{x}, \tag{14}$$

where V is the volume domain. The spectrum $\phi_{i,j}(\mathbf{k})$ is then defined by,

$$E = \frac{1}{2}\iiint \langle \phi_{i,i}(\mathbf{k}) \rangle \mathrm{d}\mathbf{k} = \iiint E(\mathbf{k}) \mathrm{d}\mathbf{k} \tag{15}$$

where $\phi_{i,j}(\mathbf{k})$ is the Fourier transform of the velocity correlation tensor $R_{i,j}(\mathbf{r})$,

$$\phi_{i,j}(\mathbf{k}) = \frac{1}{(2\pi)^3}\iiint R_{i,j}(\mathbf{r})e^{-\mathrm{i}\mathbf{k}.\mathbf{r}}\mathrm{d}\mathbf{r}, \qquad R_{i,j}(\mathbf{r}) = \frac{1}{V}\iiint u_j(\mathbf{x})u_i(\mathbf{x}+\mathbf{r})\mathrm{d}\mathbf{x}. \quad (16)$$

$R_{i,j}(\mathbf{r})$ tells us how velocities at points separated by a vector $\mathbf{r}$ are related. If we know these two point velocity correlations, we can deduce $E(\mathbf{k})$. Hence the energy spectrum has the information content of the two-point correlation.

$KE(\mathbf{k})$ contains directional information. More usually, we want to know the energy at a particular scale $k = |\mathbf{k}|$ without any interest in separating it by direction. To find $KE(k)$, we integrate over the spherical shell of radius k (in 3-dimensions),

$$E = \iiint E(\mathbf{k})\mathrm{d}\mathbf{k} = \int_0^\infty \left[\oint k^2 E(\mathbf{k})d\sigma\right]\mathrm{d}k = \int_0^\infty E(k)\mathrm{d}k, \quad (17)$$

where σ is the solid angle in wavenumber space, i.e. $\mathrm{d}\sigma = \sin\theta_1\ \mathrm{d}\theta_1\ \mathrm{d}\theta_2$. We now define the isotropic spectrum as,

$$E(k) = \oint k^2 E(\mathbf{k})\mathrm{d}\sigma = \frac{1}{2}\oint k^2\langle\phi_{i,i}(\mathbf{k})\rangle\mathrm{d}\sigma. \quad (18)$$

For isotropic velocity fields the spectrum does not depend on directions, i.e. $\langle\phi_{i,i}(\mathbf{k})\rangle = \phi_{i,i}(k)$, and we have,

$$E(k) = 2\pi k^2\phi_{i,i}(k). \quad (19)$$

Energy budget equation in spectral space

We have an equation for the evolution of the total kinetic energy E. Equally interesting is the evolution of $E(k)$, the isotropic energy at a particular wavenumber k. This will include terms which describe the transfer of energy from one scale to another, via nonlinear interactions.

To obtain such an equation we must take the Fourier transform of the non-rotating, unstratified Boussinesq equations,

$$\frac{\partial u_i}{\partial t} - \nu\frac{\partial^2 u_i}{\partial x_j^2} = -u_j\frac{\partial u_i}{\partial x_j} - \frac{1}{\rho_0}\frac{\partial p}{\partial x_i}. \quad (20)$$

The two terms on the lhs are linear and are easily transformed into Fourier space,

$$\frac{\partial}{\partial t}u_i(\mathbf{x},t) \iff \frac{\partial}{\partial t}\hat{u}_i(\mathbf{k},t), \quad (21)$$

$$\nu\frac{\partial^2}{\partial x_j^2}u_i(\mathbf{x},t) \iff -\nu k_j^2\hat{u}_i(\mathbf{k},t). \quad (22)$$

In order to convert the pressure gradient term, we first notice that taking the divergence of the Navier-Stokes equation we obtain,

$$\frac{\partial^2 p}{\partial x_i^2} = -\rho_0 \frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_i}. \tag{23}$$

Thus both terms on the rhs of Eq. (20) involve the product of velocities. The convolution theorem states that the Fourier transform of a product of two functions is given by the convolution of their Fourier transforms,

$$\frac{1}{V} \iiint u_i(\mathbf{x},t) u_j(\mathbf{x},t) e^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}} \mathrm{d}\mathbf{x} = \frac{1}{(2\pi)^3} \iiint \hat{u}_i(\mathbf{p},t) \hat{u}_j(\mathbf{q},t) \delta(\mathbf{p}+\mathbf{q}-\mathbf{k}) \mathrm{d}\mathbf{p}\mathrm{d}\mathbf{q}. \tag{24}$$

Applying the convolution terms to the terms on the rhs we get, The two terms on the lhs are linear and are easily transformed in Fourier space,

$$u_j \frac{\partial u_i}{\partial x_j} \iff \mathrm{i} \iiint q_j \hat{u}_j(\mathbf{p},t) \hat{u}_i(\mathbf{q},t) \delta(\mathbf{p}+\mathbf{q}-\mathbf{k}) \mathrm{d}\mathbf{p}\mathrm{d}\mathbf{q}, \tag{25}$$

$$p \iff \rho_0 \iiint \frac{p_i q_j}{k^2} \hat{u}_j(\mathbf{p},t) \hat{u}_i(\mathbf{q},t) \delta(\mathbf{p}+\mathbf{q}-\mathbf{k}) \mathrm{d}\mathbf{p}\mathrm{d}\mathbf{q}. \tag{26}$$

Plugging all these expressions in Eq. (20) we obtain the Navier-Stokes equation in Fourier space,

$$\left(\frac{\partial}{\partial t} + \nu k^2\right) \hat{u}_i(\mathbf{k},t) = -\mathrm{i} \iiint q_j \left(\delta_{i,m} - \frac{k_i p_m}{k^2}\right) \hat{u}_j(\mathbf{p},t) \hat{u}_m(\mathbf{q},t) \delta(\mathbf{p}+\mathbf{q}-\mathbf{k}) \mathrm{d}\mathbf{p}\mathrm{d}\mathbf{q} \tag{27}$$

The term on the right hand side shows that the nonlinear terms involve triad interactions between wave vectors such that $\mathbf{k} = \mathbf{p} + \mathbf{q}$.

Now to obtain the energy equation we multiply e. (27) by $\hat{u}_i^*(\mathbf{k},t)$ and we integrate over $\mathbf{k}$,

$$\left(\frac{\partial}{\partial t} + 2\nu k^2\right) \phi_{i,i}(\mathbf{k},t) =$$
$$\mathrm{Re}\left[\iiint A_{ijm}(\mathbf{k},\mathbf{p},\mathbf{q}) \hat{u}_i^*(\mathbf{k},t) \hat{u}_j(\mathbf{p},t) \hat{u}_m(\mathbf{q},t) \delta(\mathbf{p}+\mathbf{q}-\mathbf{k}) \mathrm{d}\mathbf{p}\mathrm{d}\mathbf{q}\mathrm{d}\mathbf{k}\right]. \tag{28}$$

The terms on the rhs represent the triad interactions that exchange energy between $\hat{u}_i(\mathbf{k},t)$, $\hat{u}_j(\mathbf{p},t)$, and $\hat{u}_m(\mathbf{q},t)$. The coefficient A_{ijm} are the coupling coefficient of each triad and depends only on the wavenumbers.

If pressure and advection were not present, the energy equation would reduce to,

$$\left(\frac{\partial}{\partial t} + 2\nu k^2\right) \phi_{i,i}(\mathbf{k},t) = 0, \tag{29}$$

in which the wavenumbers are uncoupled. The solution to this equation is,

$$\phi_{i,i}(\mathbf{k},t) = \phi_{i,i}(\mathbf{k},0)e^{-\nu k^2 t}. \tag{30}$$

According to (30), the energy in wavenumber $\mathbf{k}$ decays exponentially, at a rate that increases with increasing wavenumber magnitude k. Thus viscosity damps the smallest spatial scale fastest.

We now restrict our analysis to isotropic velocity fields, so that we can use (19) and simplify (28),

$$\frac{\partial}{\partial t}E(k,t) = T(k,t) - 2\nu k^2 E(k,t), \tag{31}$$

where $T(k,t)$ comprises all triad interaction terms. If we examine the integral of this equation over all k,

$$\frac{\partial}{\partial t}\int_0^\infty E(k)\mathrm{d}k = \int_0^\infty T(k,t)\mathrm{d}k - 2\nu \int_0^\infty k^2 E(k)\mathrm{d}k, \tag{32}$$

and note that $-2\nu k^2 E(k)$ is the Fourier transform of the dissipation term, then we see that the equation for the total energy budget in (6), is recovered only if,

$$\int_0^\infty T(k,t)\mathrm{d}k = 0. \tag{33}$$

Hence the nonlinear interactions transfer energy between different wave numbers, but do not change the total energy.

Now, adding a forcing term to the energy equation in k-space we have the following equation for energy at a particular wavenumber k,

$$\frac{\partial}{\partial t}E(k,t) = T(k,t) + F(k,t) - 2\nu k^2 E(k,t), \tag{34}$$

where $F(k,t)$ is the forcing term, and $T(k,t)$ is the **kinetic energy transfer**, due to nonlinear interactions. The **kinetic energy flux** through wave number k is $\Pi(k,t)$, defined as,

$$\Pi(k,t) = \int_k^\infty T(k',t)\mathrm{d}k', \tag{35}$$

or

$$T(k,t) = -\frac{\partial \Pi(k,t)}{\partial k}. \tag{36}$$

For stationary turbulence,

$$2\nu k^2 E(k) = T(k) + F(k). \tag{37}$$

Remembering that the total dissipation rate is given by,

$$\epsilon = \int_0^\infty 2\nu k^2 E(k)\mathrm{d}k \tag{38}$$

and that the integral of the triad interactions over the whole k-space vanishes, we have,

$$\epsilon = \int_0^\infty F(k)\mathrm{d}k. \tag{39}$$

The rate of dissipation of energy is equal to the rate of injection of energy.

If the forcing $F(k)$ is concentrated on a narrow spectral band centered around a wave number k_i, then for $k \neq k_i$,

$$2\nu k^2 E(k) = T(k). \tag{40}$$

In the limit of $\nu \to 0$, the energy dissipation becomes negligible at large scales. Thus there must be an intermediate range of scales between the forcing scale and the scale where viscous dissipation becomes important, where,

$$2\nu k^2 E(k) = T(k) \approx 0. \tag{41}$$

Notice that ϵ must remain nonzero, for nonzero $F(k)$, in order to balance the energy injection. This is achieved by $\int_0^\infty k^2 E(k)\mathrm{d}k \to \infty$, i.e. the velocity fluctuations at small scales increase.

Then we find the energy flux in the limit $\nu \to 0$,

$$\begin{aligned} \Pi(k) = 0, \; : \; k < k_i \\ \Pi(k) = \epsilon \; : \; k > k_i \end{aligned} \tag{42}$$

Hence at vanishing viscosity, the kinetic energy flux is constant and equal to the injection rate, for wavenumbers greater than the injection wavenumber k_i. The scenario is as follows. (a) Energy is input at a rate ϵ at a wavenumber k_i. (b) Energy is fluxed to higher wavenumbers at a rate ϵ trough triad interactions. (c) Energy is eventually dissipated at very high wavenumbers at a rate ϵ, even in the limit of $\nu \to 0$.

The statement that triad interactions produce a finite energy flux ϵ toward small scales does not mean that all triad interactions transfer energy exclusively toward small scales. Triad interactions transfer large amounts of energy toward both large and small scales. On average, however, there is an excess of energy transfer toward small scales given by ϵ.

Kolmogorov spectrum

Kolmogorov's 1941 theory for the energy spectrum makes use of the result that ϵ, the energy injection rate, and dissipation rate also controls the flux of energy. Energy flux is independent of wavenumber k, and equal to ϵ for $k > k_i$. Kolmogorov's theory assumes the injection wavenumber is much less than the dissipation wavenumber ($k_i << k_d$, or large Re). In the intermediate range of scales $k_i < k < k_d$ neither the forcing nor the viscosity are explicitly important, but instead the energy flux ϵ and the local wavenumber k are the only controlling parameters. Then we can express the energy density as

$$E(k) = f(\epsilon, k) \tag{43}$$

Now using dimensional analysis:

Quantity	Dimension
Wavenumber k	$1/L$
Energy per unit mass E	$U^2 \sim L^2/T^2$
Energy spectrum $E(k)$	$EL \sim L^3/T^2$
Energy flux ϵ	$E/T \sim L^2/T^3$

In Eq. (43) the lhs has dimensionality L^3/T^2; the dimension T^{-2} can only be balanced by $\epsilon^{2/3}$ because k has no time dependence. Thus,

$$E(k) = \epsilon^{2/3} g(k). \tag{44}$$

Now $g(k)$ must have dimensions $L^{5/3}$ and the functional dependence we must have, if the assumptions hold, is,

$$E(k) = C_K \epsilon^{2/3} k^{-5/3} \tag{45}$$

This is the famous *Kolmogorov spectrum*, one of the cornerstone of turbulence theory. C_K is a universal constant, the *Kolmogorov constant*, experimentally found to be approximately 1.5. The region of parameter space in k where the energy spectrum follows this $k^{-5/3}$ form is known as the **inertial range**. In this range, energy **cascades** from the larger scales where it was injected ultimately to the dissipation scale. The theory assumes that the spectrum at any particular k depends only on spectrally local quantities - i.e. has no dependence on k_i for example. Hence the possibility for long-range interactions is ignored.

We can also derive the Kolmogorov spectrum in a perhaps more physical way (after Obukhov). Define an eddy turnover time $\tau(k)$ at wavenumber k as the time taken for a parcel with energy $E(k)$ to move a distance $1/k$. If $\tau(k)$ depends only on $E(k)$ and k then, from dimensional analysis,

$$\tau(k) \sim \left[k^3 E(k)\right]^{-1/2} \tag{46}$$

The energy flux can be defined as the available energy divided by the characteristic time τ. The available energy at a wavenumber k is of the order of $kE(k)$. Then we have,

$$\epsilon \sim \frac{kE(k)}{\tau(k)} \sim k^{5/2} E(k)^{3/2}, \tag{47}$$

and hence,

$$E(k) \sim \epsilon^{2/3} k^{-5/3}. \tag{48}$$

Characteristic scales of turbulence

Kolmogorov scale

We have shown that viscous dissipation acts most efficiently at small scales. Thus above a certain wavenumber k_d, viscosity will become important, and $E(k)$ will decay more rapidly than in the inertial range. The regime $k > k_d$ is known as the *dissipation range*. A simple scaling argument for k_d can be made by assuming that the spectrum follows the inertial scaling until k_d and then drops suddenly to zero because of viscous dissipation. In reality the transition between the two regimes is more gradual, but this simple model predicts k_d quite accurately. First we assume,

$$\begin{aligned} E(k) &= C_K \epsilon^{2/3} k^{-5/3}, \qquad k_i < k < k_d, \\ E(k) &= 0, \qquad k > k_d. \end{aligned} \tag{49}$$

Substituting (38), and integrating between k_i and k_d we find,

$$k_d \sim \left(\frac{\epsilon^{1/4}}{\nu^{3/4}} \right). \tag{50}$$

The inverse $l_d = 1/k_d$ is known as the *Kolmogorov scale*, the scale at which dissipation becomes important.

$$l_d \sim \left(\frac{\nu^{3/4}}{\epsilon^{1/4}} \right) \tag{51}$$

Integral scale

At the small wavenumber end of the spectrum, the important lengthscale is l_i, the integral scale, the scale of the energy-containing eddies. $l_i = 1/k_i$. We can evaluate l_i in terms of ϵ. Let us write,

$$U^2 = 2 \int_0^\infty E(k) \mathrm{d}k \tag{52}$$

and substituting for $E(k)$ from (45),

$$U^2 \sim 2 \int_0^\infty C_K \epsilon^{2/3} k^{-5/3} \mathrm{d}k \sim 3 C_K \epsilon^{2/3} k_i^{-2/3}. \tag{53}$$

Then,

$$k_i \sim \frac{\epsilon}{U^3} \tag{54}$$

so that $l_i \sim U^3/\epsilon$. Then the ratio of maximum and minimum dynamically active scales,

$$\frac{l_i}{l_d} = \frac{k_d}{k_i} \sim \frac{U^3}{\epsilon^{3/4} \nu^{3/4}} \sim \left(\frac{U l_i}{\nu} \right)^{3/4} \sim Re_{l_i}^{3/4}. \tag{55}$$

where Re_{l_i} is the *integral Reynolds number*. Hence in K41 the inertial range spans a range of scales growing as the (3/4)th power of the integral Reynolds number. It follows that if we want to describe such a flow accurately in a numerical simulation on a uniform grid, the minimum number of points per integral scale is $N \sim Re_{l_i}^{9/4}$. One consequence is that the storage requirements of numerical simulations scale as $Re_{l_i}^{9/4}$. Since the time step has usually to be taken proportional to the spatial mesh, the total computational work needed to integrate the equations for a fixed number of large eddy turnover times grows as $Re_{l_i}^3$. This shows that progress in achieving high Re_{l_i} simulations is very slow.

3D homogeneous isotropic turbulence in geophysical flows

The assumptions of homogeneity, stationarity and isotropy as employed by Taylor and Kolmogorov have permitted tremendous advances in our understanding of turbulence. In particular the theory of Kolmogorov remains an outstanding example of what we mean by emergence of statistical predictability in a chaotic system. However this a a course on geophysical turbulence and we must ultimately confront the fact that real world physical flows rarely conform to our simplifying assumptions. In geophysical turbulence, statistical symmetries are upset by a complex interplay of effects. Here we focus on three important class of phenomena that modify small-scale turbulence in the atmosphere and ocean: large-scale shear, stratification, and boundary proximity. In a few weeks we will consider the role of rotation.

A very nice description of the effect of shear, stratification, and boundary proximity on small-scale geophysical turbulence is given in the review article on "3D Turbulence" by Smyth and Moum included as part of the reading material. The student should read that paper before proceeding with this chapter. Here we only provide some additional comments on the definition of the Ozimdov scale.

Ozmidov scale

In geophysical flows 3D turbulence can be a reasonable approximation at scales small enough that buoyancy and rotation effects can be neglected. Stratification becomes important at scales smaller than rotation and it is therefore more important in setting the upper scales at which 3D arguments hold. Stratification affects turbulence when the Froude number $Fr = U/(NH) < 1$, where U is a typical velocity scale, and H a typical vertical length scale of the motion. For large Fr, the kinetic energy of the motion is much larger than the potential energy changes involved in making vertical excursions of order H. For small Fr, the stratification suppresses the vertical motion because a substantial fraction of kinetic energy must be converted to potential energy when a parcel moves in the vertical.

We can define a characteristic scale l_B at which overturning is suppressed by the buoyancy stratification as follows. The velocity associated with a particular length scale l in high Reynolds number isotropic 3D turbulence scales like,

$$u^2 \sim \epsilon^{2/3} k^{-2/3} \qquad \Longleftrightarrow \qquad u \sim (l\epsilon)^{1/3}. \tag{56}$$

Vertical motion at length scale l will be suppressed by the stratification when the local Froude number $Fr_l = 1$. If we define the length scale at which this suppression occurs as l_B then,

$$\frac{u_B}{Nl_B} = \frac{(l_B\epsilon)^{1/3}}{Nl_B} = 1 \qquad \Longrightarrow \qquad l_B = \left(\frac{\epsilon}{N^3}\right)^{1/2} \tag{57}$$

where l_B is known as the *Ozmidov scale.*

In stratified geophysical flows, we have a scenario in which a regime transition occurs at l_B.

- $l < l_B$: Fully 3D, isotropic turbulence. In this regime stratification can be neglected, and an inertial range may exist, if $l_d << l_B$, i.e. $\epsilon/(\nu N^2) >> 1$.
- $l > l_B$: Stratification influenced regime. In this regime ϵ is no longer constant with wave number, since some kinetic energy is lost through conversion to potential energy. 3D turbulence is replaced by motion controlled by the buoyancy stratification: either internal waves, or a quasi-2-dimensional turbulence, often described as "pancake turbulence", characterized by strong vortical motions in decoupled horizontal layers.

(For more on stratified turbulence, see Lesieur Ch XIII, Metais and Herring, 1989: Numerical simulations of freely evolving turbulence in stably stratified fluids. *J. Fluid Mech.*, **239**. Fincham, Maxworthy and Spedding, 1996: Energy dissipation and vortex structure in freely-decaying stratified grid turbulence. *Dyn. Atmos. Oceans*, **23**, 155-169.)

Further reading: Lesieur, Ch V, VI; Tennekes and Lumley, Ch 8; Frisch, Ch 5, 6, 7, 8.

Intermittency in isotropic 3D turbulence

[This chapter is a synthesis of material taken from Frisch (Turbulence, Cambridge University Press, 1995) and Salmon (Lectures on Geophysical Fluid Dynamics, Oxford University Press, 1998)]

Kolmogorov's 1941 model for the energy spectrum of isotropic homogeneous 3D turbulence assumes that there is an intermediate region in wavenumber space, the *inertial range*, where neither the forcing nor the viscosity are explicitly important. This simple assumption paves the road to determining the shape of the energy spectrum. Apart from forcing and dissipation, there are only two other dimensional parameters in the Navier-Stokes equations: the energy flux ϵ and the local wavenumber k. Thus the energy spectrum must be a function of these two parameters only. Dimensional consistency is all that is required to get an expression for the energy spectrum,

$$E(k) \approx C_K \epsilon^{2/3} k^{-5/3}. \tag{58}$$

The model assumes that the spectra at any particular k depends only on spectrally local quantities: the possibility for long-range interactions is ignored. In this chapter we discuss whether the assumption of locality is satisfied in real flows, and we examine some of the developments in the theory of 3D turbulence beyond Kolmogorov's seminal work.

Landau and the lack of universality in turbulence

In a famous footnote in his book on fluid dynamics, L.D. Landau noted an important inconsistency in K41 and objected to its universality. This led to a revision of the theory, but most people feel that it also destroyed the hope that there can be an exact theory. Landau's objection is neither the only, nor the most serious objection to K41. However it has helped the scientific community to better appreciate the enormous assumptions underlying Kolmogorov's theory.

Landau's remark appeared in a footnote in the 1944 edition of his book on *Fluid mechanics*, but in later editions found its way into the main text. Here is the full text of the remark, as it appears on page 140 of the second edition of the English translation of the book. The only changes are the substitution of Landau's notation, with the notation used in these notes.

One further general remark should be made. It might be thought that the possibility exist in principle of obtaining a universal formula, applicable to any turbulent flow, which should give $S_2(r)$ for all r that are small compared to r_0. In fact, however, there can be no such formula, as we see from the following argument. The instantaneous value of $(\delta v(r))^2$ might in principle be expressed as a universal function of the energy dissipation ϵ at the instant considered. When we average these expressions, however, an important part will be played by the manner of variation of ϵ over times of the order of periods of the large eddies (with size $\sim r_0$), and this variation is different for different flows. The result of the averaging therefore cannot be universal.

Kraichnan (1974) gave an illuminating reformulation of Landau's footnote remark. The essence of Landau's objection is that K41 cannot apply to a collection of flows with different dissipation rates ϵ. First consider two completely separate flows. The first flow is vigorously stirred so that ϵ_1 is large. The second flow is weakly stirred so that ϵ_2 is small. If both flows are fully turbulent, then according to K41,

$$E_1(k) = C_K \epsilon_1^{2/3} k^{-5/3}, \qquad \text{and} \qquad E_2(k) = C_K \epsilon_2^{2/3} k^{-5/3}. \tag{59}$$

Next consider a system composed of these two separate flows. If the flows occupy equal volumes, then the dissipation and the energy spectrum of the composite system are given by,

$$\epsilon = \frac{1}{2}\left(\epsilon_1 + \epsilon_2\right), \qquad \text{and} \qquad E(k) = \frac{1}{2}\left(E_1(k) + E_2(k)\right). \tag{60}$$

Thus for the composite system,

$$E(k) \neq C_K \epsilon^{2/3} k^{-5/3}. \tag{61}$$

That is, the composite system does not obey K41, essentially because the average of a two-thirds power is not equal to the two-thirds power of the average.

So far there seems to be no problem, because the composite flow is not a single flow, and hence there is no reason why K41 should apply to it. But suppose that the subscripts 1 and 2 do not refer to two flows, but to two large regions of the same flow with locally different dissipation rates. We conclude that K41 fails in cases where the dissipation rate ϵ, averaged over length scales characteristic of the inertial range, fluctuates.

Frisch in chapter 8 of his book on *Turbulence* shows two examples of irregular signals. The signal in Figure 8.1 is *self-similar*, *i.e.* successive enlargements of the signal have the same general aspect, regardless of where the magnification window is positioned. The signal in Figure 8.2 is *intermittent*, *i.e.* it displays activity during only a fraction of the time, which decreases with the scale under consideration. Enlargements of different sections of the signal produce completely different results, depending on whether the window is positioned on an active or passive period. When dealing with intermittent signals, the smaller the window, the more carefully it must be positioned to produce a nontrivial function.

The model of Kolmogorov relied on the assumption that turbulent signals are self-similar. Landau pointed out that, if dissipation is intermittent, then the model K41 had to be reconsidered. Laboratory experiments showed that Landau's remark was right on the spot: dissipation signals are strongly intermittent. In this section, we will discuss the theoretical arguments that have been proposed to reconcile Kolmogorov and Landau.

Coherent structures

Half a century after Kolmogorov's work on the statistical theory of three dimensional turbulence, we still wonder how his work can be reconciled with Leonardo's half a millennium old drawings of eddy motion in the study for the elimination of rapids in the river Arno. Indeed, Kolmogorov's work on turbulence, ignores any structure which may be present in the flow.

In the first lecture, we pointed out that many turbulent flows are known to possess *coherent structures*. Their rediscovery by Crow and Champagne (1971) and Brown and Roshko (1974) has led to questioning the relevance of the traditional statistical theory of turbulence. The accepted paradigm is that, as far as the inertial-range properties are concerned, coherent structures do not matter if they are confined to the large scales of the flow. But is this really the case? And is there a fully developed inertial range in geophysical turbulence, where inhomogeneities and coherent structures do not appear?

A nice discussion of the dichotomy between the spectral description of turbulence and the description based on coherent patterns in real space can be found in the paper by Armi and Flament (Journal of Geophysical Research, 90, 1985).

Further reading: Frisch, Ch 7, 8.