# Rotating Shallow-Water Waves

We now consider the effects of rotation and boundaries on fluids obeying (in the horizontal) the linearized shallow-water equations

$$\frac{\partial}{\partial t}\mathbf{u} + f\hat{\mathbf{z}} \times \mathbf{u} = -\nabla\phi$$
$$\frac{\partial}{\partial t}\phi + gH\nabla \cdot \mathbf{u} = 0$$

where $\mathbf{u}$ and $\nabla$ are horizontal vectors/ operators.

**Plane waves**

For the simplest case, we take all fields proportional to $\exp(\imath\mathbf{k} \cdot \mathbf{x} - \imath\omega t)$ to find

$$\begin{pmatrix} -\imath\omega & -f & \imath k \\ f & -\imath\omega & \imath\ell \\ gH\imath k & gH\imath\ell & -\imath\omega \end{pmatrix} \begin{pmatrix} u \\ v \\ \phi \end{pmatrix} = 0$$

which implies

$$\imath\omega[\omega^2 - f^2 - gH(k^2 + \ell^2)] = 0$$

which has three roots, $\omega = 0$ and

$$\omega^2 = f^2 + gH|\mathbf{k}|^2$$

which is the generalization of the long gravity wave dispersion relation. In the presence of rotation, the waves become dispersive, with

$$\mathbf{c}_g = \sqrt{gH}\frac{\mathbf{k}}{\sqrt{\frac{f^2}{gH} + |\mathbf{k}|^2}}$$

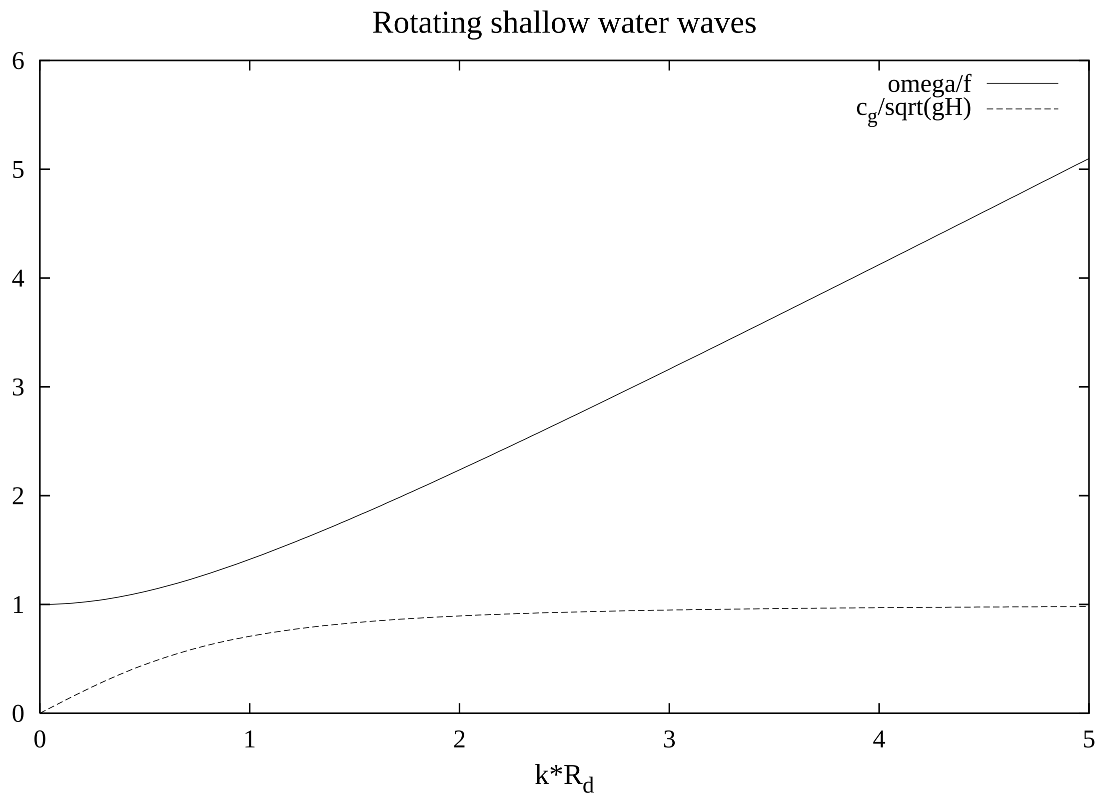
These can be simplified by using the deformation radius $\sqrt{gH}/f$ as a length scale

$$\frac{\omega}{f} = \sqrt{1 + |\mathbf{k}R_d|^2} \quad , \quad \frac{c_g}{\sqrt{gH}} = \frac{\mathbf{k}R_d}{\sqrt{1 + |\mathbf{k}R_d|^2}}$$

Note that the shallow water equations will only be applicable for

$$kH = kR_d\frac{H}{R_d} << 1 \quad \Rightarrow \quad kR_d << \frac{\sqrt{g/H}}{f} \sim 500$$

for a $4000m$ deep ocean.




Dispersion relation for rotating plane waves

The $\omega = 0$ root is non-trivial; to see this, let us look at the equations in vorticity/divergence form. If $\zeta = \hat{\mathbf{z}} \cdot (\nabla \times \mathbf{u}) = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$ and $D = \cdot\nabla\mathbf{u}$, then

$$\frac{\partial}{\partial t}\zeta + fD = 0$$
$$\frac{\partial}{\partial t}D - f\zeta = -\nabla^2\phi$$
$$\frac{\partial}{\partial t}\phi + gHD = 0$$

Eliminating $D$ from the first and third equation gives

$$\frac{\partial}{\partial t}\left(\zeta - f\frac{\phi}{gH}\right) \equiv \frac{\partial}{\partial t}q = 0$$

The (linearized) potential vorticity $q = \left(\zeta - f\frac{\phi}{gH}\right)$ is conserved. This equation implies either the frequency is zero and the potential vorticity is not, or vice-versa. The zero-frequency waves correspond to $D = 0$, $\zeta = \nabla^2\phi/f$ and are geostrophically balanced. When $f$ varies, these turn into Rossby waves.

If we recast the divergence equation in terms of $q$

$$\frac{\partial}{\partial t}D - fq - \frac{f^2}{gH}\phi = -\nabla^2\phi$$

and use the conservation of mass equation, we find

$$\frac{\partial^2}{\partial t^2}\phi + fgHq + f^2\phi = gH\nabla^2\phi \tag{1}$$

For the gravity waves with no PV signal,

$$\frac{\partial^2}{\partial t^2}\phi + f^2\phi = gH\nabla^2\phi \tag{2}$$

and we recover the dispersion relation above.

**Adjustment**

If we consider the initial value problem, we can specify the three fields, or, alternatively, we can specify $q(\mathbf{x})$, $\phi(\mathbf{x},0) = \phi_0(\mathbf{x})$, and $\frac{\partial}{\partial t}\phi(\mathbf{x},0) = \phi_{t0}(\mathbf{x})$. Since $q$ remains unchanged, we can split the pressure up into the geostrophic part and the gravity wave part

$$\phi = \phi_g(\mathbf{x}) + \phi_w(\mathbf{x},t)$$

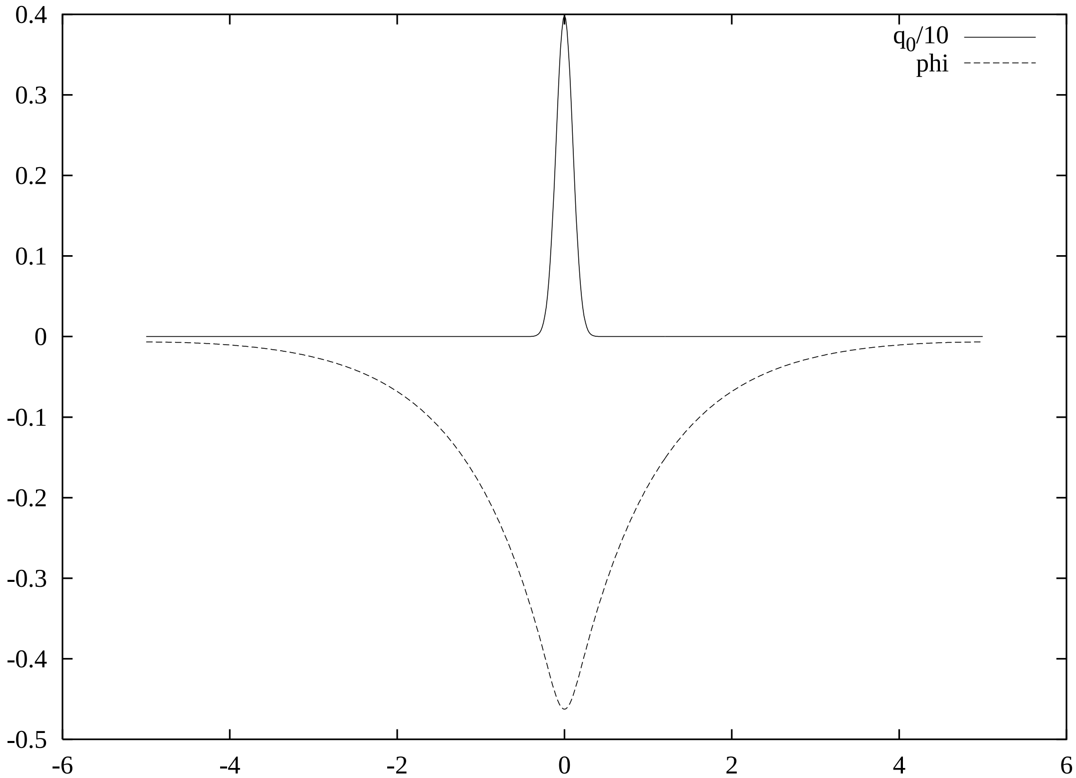
$$\nabla^2\phi_g - \frac{f^2}{gh}\phi_g = \frac{1}{f}q$$

$$\frac{\partial^2}{\partial t^2}\phi_w + f^2\phi_w = gH\nabla^2\phi_w$$

$$\phi_w(\mathbf{x},0) = \phi_0(\mathbf{x}) - \phi_g(\mathbf{x}) \quad , \quad \frac{\partial}{\partial t}\phi_w(\mathbf{x},0) = \phi_{t0}(\mathbf{x})$$

The equation for the geostrophic pressure shows that the influence of a localized potential vorticity anomaly spreads out over a scale $R_d = \sqrt{gH}/f$ called the "deformation radius." I.e., if $q = q_0\delta(x)$ (independent of $y$), the geostophic pressure is

$$\phi_g = -\frac{q_0 R_d}{2f}\exp(-|x/R_d|)$$




Initial PV and adjusted $\phi_g$ state

### Gravity waves in a channel

Now we consider waves in a channel $0 < y < W$. In that case, we must apply the boundary conditions $v = 0$ at $y = 0, W$. For the non-rotating case, the $y$-momentum equation implies $\frac{\partial}{\partial y}\phi = 0$ at the boundaries (or, more generally, $\nabla\phi \cdot \hat{\mathbf{n}} = 0$). The solutions to the $f = 0$ version of (2) are

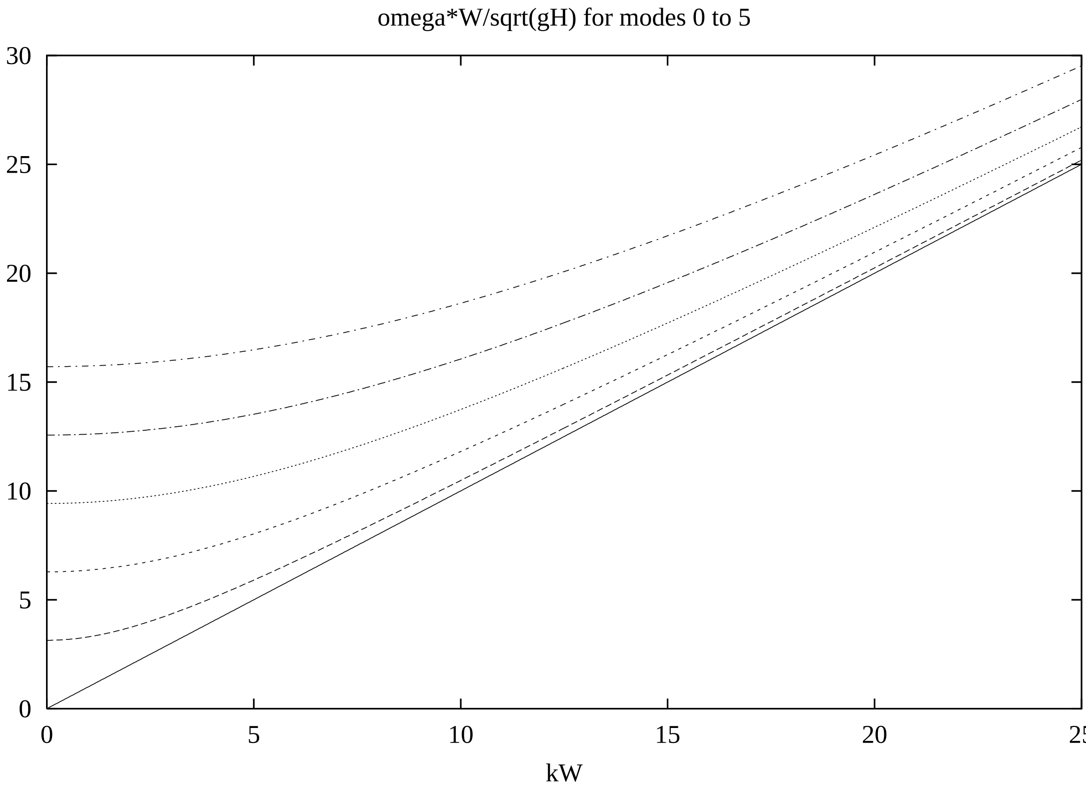
$$\phi = \cos(\ell y) e^{\imath(kx - \omega t)}$$

with

$$\omega^2 = gH(k^2 + \ell^2)$$

and

$$\ell = 0,\ \frac{\pi}{W},\ \frac{2\pi}{W},\ \frac{3\pi}{W} \ldots$$




Frequencies $\omega W/\sqrt{gH}$ for $\ell W = 0, 1, 2, 3, 4, 5$

The rotating case is more complex. If we stick with equation (2), we can use the two momentum equations with $v = 0$ to show that

$$\frac{\partial}{\partial t}u = -\frac{\partial}{\partial x}\phi \quad , \quad fu = -\frac{\partial}{\partial y}\phi \quad \Rightarrow \quad \frac{\partial^2 \phi}{\partial t \partial y} = f\frac{\partial \phi}{\partial x}$$

For waves with $\phi = \Phi(y)\exp(\imath kx - \imath\omega t)$, we must satisfy

$$\frac{\partial}{\partial y}\Phi = -\frac{fk}{\omega}\Phi \quad at \quad y = 0,\ W$$

and

$$gH\frac{\partial^2}{\partial y^2}\Phi = (f^2 + gHk^2 - \omega^2)\Phi$$

We can look for solutions $\Phi = \cos(\ell y + \theta)$; the dispersion relation is then the same as for plane waves, but the boundary conditions imply

$$\ell \sin\theta = \frac{fk}{\omega}\cos\theta \quad and \quad \ell\sin(\ell W + \theta) = \frac{fk}{\omega}\cos(\ell W + \theta) \quad \Rightarrow \quad \tan(\theta) = \tan(\ell W + \theta)$$

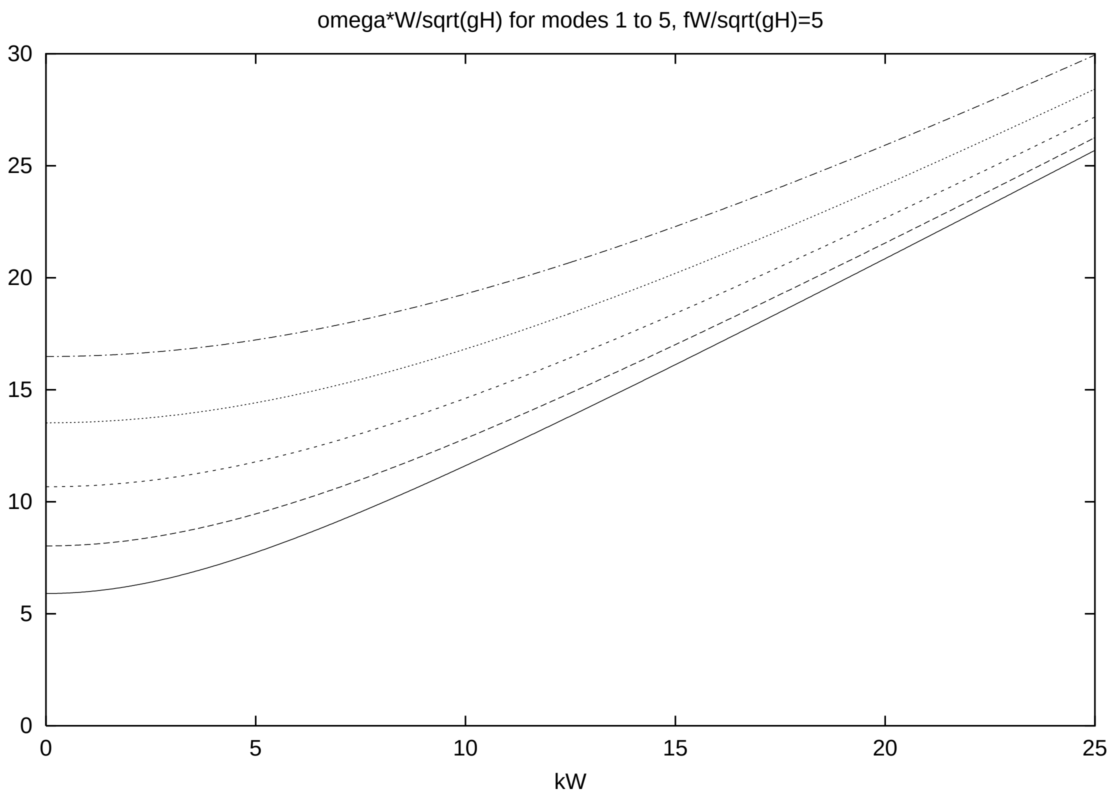
Thus $\ell W$ is still an integer multiple of $\pi$. However, the $\ell = 0$ solution is no longer satisfactory, since it makes $\Phi$ constant, which will not be consistent with the boundary conditions.

The dispersion relation

$$\omega^2 = f^2 + gH(k^2 + \ell^2) = f^2 + gH(k^2 + \frac{n^2\pi^2}{W^2})$$

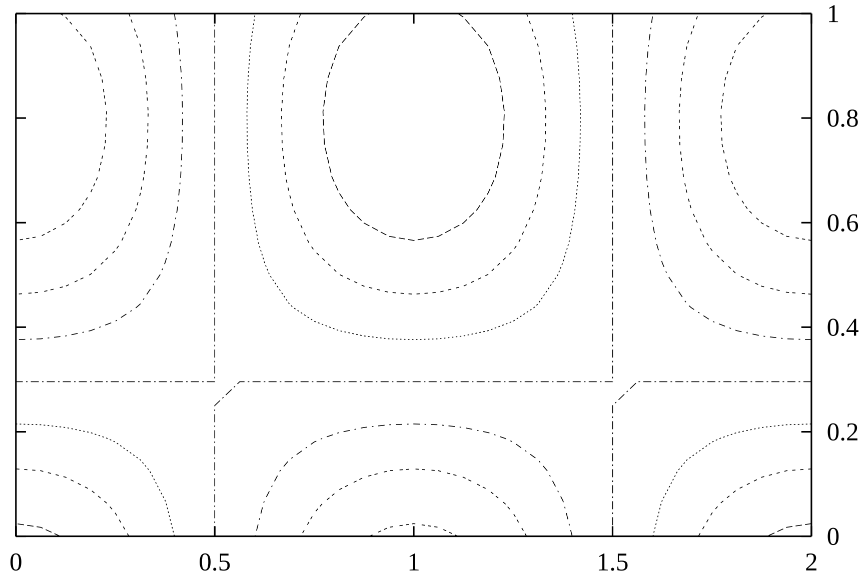
now correspond to modes with the same cross channel wavelength as before, but which no longer have their maxima at the channel walls:

$$\tan\theta = \frac{fk}{\ell\omega}$$




Dispersion relation in rotating channel

A sample waveform for $kW = \pi$, $\ell W = \pi$, $f/\sqrt(gH) = 5$ is




Sample waveform

**Kelvin waves**

But we can also look for exponential solutions; we can see that

$$\Phi = \exp\left(-\frac{fk}{\omega}y\right)$$

clearly satisfies the boundary conditions. Starting with the general case $Ae^{\alpha y} + Be^{-\alpha y}$ leads to the conclusion that the solution above is the only correct one. Putting this into the equation for $\Phi$ gives

$$gH\frac{f^2k^2}{\omega^2} = f^2 + gHk^2 - \omega^2$$

which has the solutions

$$\omega^2 = gHk^2 \quad , \quad \omega^2 = f^2$$

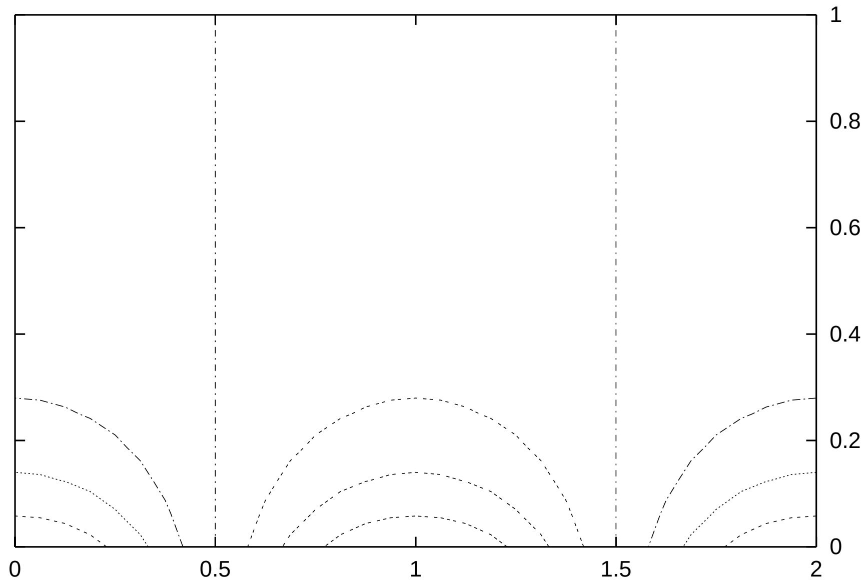
The latter is spurious; if we examine the momentum equations with $\omega = f$, we find $u = \frac{k}{f}\phi$ since the other solution $v = \imath\frac{k}{f}\phi$ will not satisfy the boundary conditions. The mass equation then implies $f^2 = gHk^2$ which is not generally correct. Thus, we find that the $\ell = 0$ mode is replaced by one which decays across the channel as

$$\Phi = \exp\left(-\frac{f}{\sqrt{gH}}y\right) = \exp\left(-\frac{y}{R_d}\right)$$

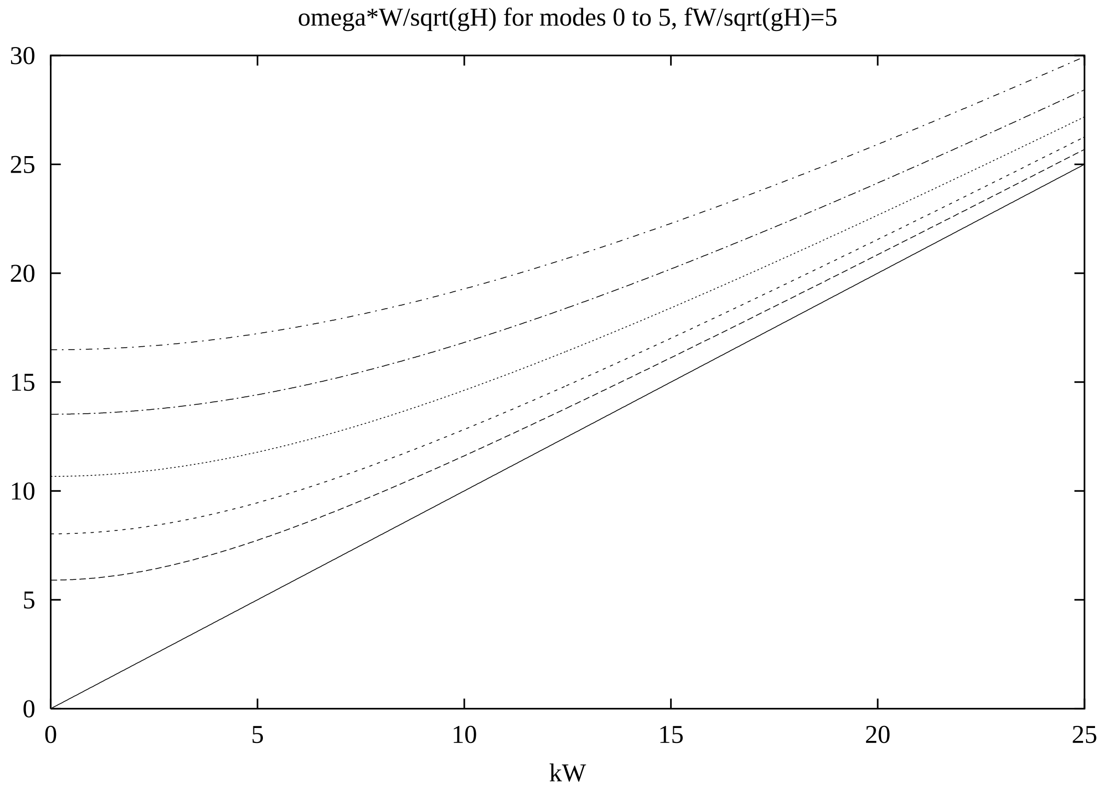
and has frequency

$$\omega = \sqrt{gH}k$$

These non-dispersive waves are called Kelvin waves.




Kelvin mode




Full dispersion relation

The Kelvin wave dynamics is also simple. The $v$ field would have the same cross-channel structure but can only match the boundary conditions if it is everywhere zero. Therefore, we have

$$\frac{\partial}{\partial t}u = -\frac{\partial}{\partial x}\phi$$
$$-fu = -\frac{\partial}{\partial y}\phi$$
$$\frac{\partial}{\partial t}\phi + gH\frac{\partial}{\partial x}u = 0$$

The $1^{st}$ and $3^{rd}$ equations look like non-rotating gravity waves; the second shows that the cross-stream balance is geostrophic – the pressure varies across the channel in a way consistent with the Coriolis force on the along-channel flow.