LTE Hydrostatic version

We begin with the linearized, traditional approx. eqns. with the hydrostatic approx.

$$\frac{\partial}{\partial t}\mathbf{u} + \mathbf{f} \times \mathbf{u} = -\frac{1}{\overline{\rho}}\nabla p'$$
$$\frac{\partial}{\partial z}p' = -g\rho'$$
$$\frac{1}{\overline{\rho}}\frac{\partial}{\partial t}\rho' + \frac{1}{\overline{\rho}}\frac{\partial}{\partial z}\overline{\rho}w + \nabla \cdot \mathbf{u} = 0$$
$$\frac{\partial}{\partial t}\rho' - \frac{1}{c_s^2}\frac{\partial}{\partial t}p' + w\left[\frac{\partial \overline{\rho}}{\partial z} - \frac{g\overline{\rho}}{c_s^2}\right] = 0$$

(with c_s^2 defined in terms of $\overline{\rho}$ and $\overline{p}$ and being only a function of z). We define the dynamic pressure by $p' = \overline{\rho}\phi$ so that the horizontal eqns become the same as the SW eqns:

$$\boxed{\frac{\partial}{\partial t}\mathbf{u} + \mathbf{f} \times \mathbf{u} = -\nabla\phi} \tag{1}$$

We multiply the thermodynamic equation by $g/\overline{\rho}$ to get

$$\frac{\partial}{\partial t}\left[\frac{g\rho'}{\overline{\rho}} - \frac{gp'}{\overline{\rho}c_s^2}\right] = wN^2$$

so that

$$wN^2 = \frac{\partial}{\partial t}\left[-\frac{1}{\overline{\rho}}\frac{\partial}{\partial z}\overline{\rho}\phi - \frac{g}{c_s^2}\phi\right] = \frac{N^2}{g}\frac{\partial}{\partial t}\phi - \frac{\partial}{\partial t}\frac{\partial \phi}{\partial z}$$

Therefore the vertical velocity is

$$w = \frac{1}{g}\frac{\partial}{\partial t}\phi - \frac{\partial}{\partial t}\frac{1}{N^2}\frac{\partial \phi}{\partial z}$$

The conservation of mass equation becomes

$$\frac{\partial}{\partial t}\left[-\frac{1}{\overline{\rho}g}\frac{\partial \overline{\rho}\phi}{\partial z} + \frac{1}{\overline{\rho}g}\frac{\partial \overline{\rho}\phi}{\partial z} - \frac{1}{\overline{\rho}}\frac{\partial}{\partial z}\frac{\overline{\rho}}{N^2}\frac{\partial \phi}{\partial z}\right] + \nabla \cdot \mathbf{u} = 0$$

or

$$\boxed{\frac{\partial}{\partial t}\left[-\frac{1}{\overline{\rho}}\frac{\partial}{\partial z}\frac{\overline{\rho}}{N^2}\frac{\partial \phi}{\partial z}\right] + \nabla \cdot \mathbf{u} = 0}$$

We now separate variables in vertical and horizontal/time; the vertical structure satisfies

$$\frac{1}{\overline{\rho}}\frac{\partial}{\partial z}\frac{\overline{\rho}}{N^2}\frac{\partial F}{\partial z} = -\frac{1}{gH_e}F$$

and the mass equation turns into

$$\boxed{\frac{\partial}{\partial t}\phi + gH_e\nabla\cdot\mathbf{u} = 0 \qquad (2)}$$

with the terms now just representing the horizontal and temporal structure.

Equations (1) and (2) are the SW eqns on the sphere – the Laplace tidal equations.

Alternatively, we can separate at the beginning:

$$\mathbf{u} \to \mathbf{u}F(z) \quad , \quad p' \to \phi\overline{\rho}F$$

and find

$$\rho' = -\phi\frac{1}{g}\frac{\partial}{\partial z}\overline{\rho}F$$

The thermodynamics gives (following similar steps)

$$w = \left[\frac{F}{g} - \frac{1}{N^2}\frac{\partial F}{\partial z}\right]\frac{\partial\phi}{\partial t}$$

and the mass equation becomes

$$\left[-\frac{1}{\overline{\rho}}\frac{\partial}{\partial z}\frac{\overline{\rho}}{N^2}\frac{\partial F}{\partial z}\right]\frac{\partial\phi}{\partial t} + F\nabla\cdot\mathbf{u} = 0$$

giving again

$$\frac{1}{\overline{\rho}}\frac{\partial}{\partial z}\frac{\overline{\rho}}{N^2}\frac{\partial F}{\partial z} = -\frac{1}{gH_e}F$$

and (2).