Tides and Basins

Consider a basin with depth $H(\mathbf{x})$ force by an equilibrium tide $\phi_e = g\eta_e$

$$\frac{\partial}{\partial t}\mathbf{u} + f\hat{\mathbf{z}} \times \mathbf{u} = -\nabla(\phi - \phi_e)$$
$$\frac{\partial}{\partial t}\phi + \nabla \cdot [gH(\mathbf{x})\mathbf{u}] = 0$$

Let's solve for the velocities using

$$\begin{aligned}
&\mathbf{u}_t + f\hat{\mathbf{z}} \times \mathbf{u} = -\nabla(\phi - \phi_e) \\
&\qquad\qquad \text{(cross with) } \hat{\mathbf{z}} \;\Rightarrow \\
&\hat{\mathbf{z}} \times \mathbf{u}_t - f\mathbf{u} = -\hat{\mathbf{z}} \times \nabla(\phi - \phi_e) \\
&\qquad\qquad (t \text{ derivative}) \;\Rightarrow \\
&\mathbf{u}_{tt} + f^2\mathbf{u} = f\hat{\mathbf{z}} \times \nabla(\phi - \phi_e) - \frac{\partial}{\partial t}\nabla(\phi - \phi_e)
\end{aligned}$$

Assuming $e^{-\imath\omega t}$,

$$(f^2 - \omega^2)\mathbf{u} = f\hat{\mathbf{z}} \times \nabla(\phi - \phi_e) + \imath\omega\nabla(\phi - \phi_e)$$

From $\hat{\mathbf{n}}$ dot this, we get the boundary conditions

$$-f\hat{\mathbf{t}} \cdot \nabla(\phi - \phi_e) + \imath\omega\hat{\mathbf{n}} \cdot \nabla(\phi - \phi_e) = 0$$

and the dynamical equation (multiplying the mass equation by $f^2 - \omega^2$)

$$\begin{aligned}
\imath\omega(f^2 - \omega^2)\phi &= f\nabla \cdot [gH\hat{\mathbf{z}} \times \nabla(\phi - \phi_e)] + \imath\omega\nabla \cdot [gH\nabla(\phi - \phi_e)] \\
&= f\hat{\mathbf{z}} \cdot [\nabla(\phi - \phi_e) \times \nabla gH] + \imath\omega\nabla \cdot [gH\nabla(\phi - \phi_e)]
\end{aligned}$$

so the general problem is

$$\boxed{\begin{aligned}
\nabla \cdot [gH\nabla(\phi - \phi_e)] &= (f^2 - \omega^2)\phi + \frac{\imath f}{\omega}\hat{\mathbf{z}} \cdot [\nabla(\phi - \phi_e) \times \nabla gH] \\
f\hat{\mathbf{t}} \cdot \nabla(\phi - \phi_e) &= \imath\omega\hat{\mathbf{n}} \cdot \nabla(\phi - \phi_e)
\end{aligned}}$$

If we work in a circular basin with $H = H(r)$ and take an $e^{\imath m\theta}$ form, this simplifies to

$$\boxed{\begin{aligned}
\nabla \cdot gH\nabla(\phi - \phi_e) &= (f^2 - \omega^2)\phi + \frac{fm}{\omega a}gH_r(\phi - \phi_e) \\
\frac{\partial}{\partial r}(\phi - \phi_e) &= \frac{fm}{\omega a}(\phi - \phi_e)
\end{aligned}}$$

Numerical note: we would probably want to solve in terms of the height anomaly $\eta = \phi - \phi_e$ which has homogeneous but mixed boundary conditions and satisfies a forced equation in the interior.

Constant depth

When gH is constant, we have

$$\nabla^2(\phi - \phi_e) = \frac{f^2 - \omega^2}{gH}\phi$$
$$\frac{\partial}{\partial r}(\phi - \phi_e) = \frac{fm}{\omega a}(\phi - \phi_e)$$

We can write the equations in non-dimensional form, scaling distances by the domain scale a, and ω by f. Then

$$\boxed{\begin{aligned} \nabla^2(\phi - \phi_e) &= \alpha^2(1 - \omega^2)\phi \\ \frac{\partial}{\partial r}(\phi - \phi_e) &= \frac{m}{\omega}(\phi - \phi_e) \end{aligned}}$$

with $\alpha = a/R_d$ and $R_d = \sqrt{gH}/f$.

FREE MODES: when $\phi_e = 0$, we have an eigenvalue problem for ω. Let us make the *ansatz* that $\omega > 1$ (frequency greater than f – super-inertial). Then

$$\phi = J_m(Kr) \quad , \quad K^2 = \alpha^2(\omega^2 - 1)$$

This gives one equation relating K to ω; the other is

$$KJ_m'(K) = \frac{m}{\omega a}J_m(K) \quad \text{or} \quad \frac{KJ_m'(K)}{J_m(K)} = \frac{m}{\omega}$$

For $\omega >> 1$, the boundary condition says the K is a zero of J_m'; the "free-end" version of the drumhead modes (i.e., $\phi_r \simeq 0$ rather than $\phi = 0$). We expect this to happen for small basins since the time for a gravity wave to cross $a/\sqrt{gH}$ becomes short compared to $1/f$. The dispersion relation

$$\omega = \pm\sqrt{1 + (K/\alpha)}$$

has a large parameter multiplying K (which itself is greater than 2.4)

For the low frequency case,

$$\phi = I_m(Kr) \quad , \quad \omega^2 = 1 - \frac{K^2}{\alpha^2} \quad \text{and} \quad \frac{KI_m'(K)}{I_m(K)} = \frac{m}{\omega}$$

For the previous case, ω could have both positive and negatve values since the J_m functions are oscilltory. Now, however, I_m is strictly increasing so that the left-hand size is positive and ω will be positive. The waves, will propagate counter-clockwise around the basin.

They are Kelvin waves with the boundary on their right in the northern hemisphere. If ω is small enough, K will be large and $I'_m/I_M \to 1$; i.e.,

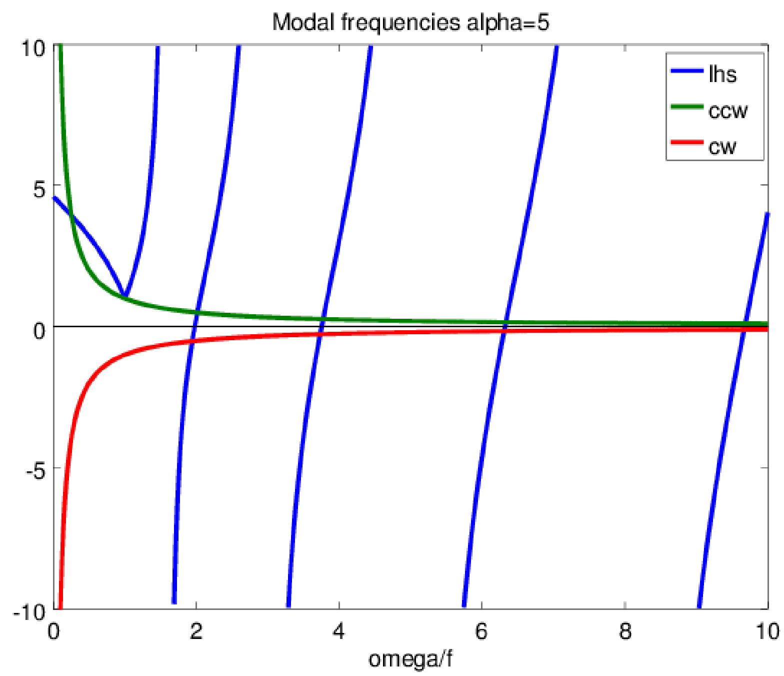
$$\omega \simeq m/K \simeq m/\alpha$$

In dimensional form, this is

$$\omega = fmR_d/a = (m/a)\sqrt{gH}$$

making the Kelvin wave characteristic clear, since m/a is the wavenumber.

We show the left and right sides of the boundary condition equations for $m = 1$;



the frequences are given by the intersections with either the clockwise ($w < 0$) or counterclockwise ($w > 0$) curves. If $\omega < f$, we have just the single ccw Kelvin mode; above f, the two Poincarè waves have quite similar frequencies. The successive roots correspond to more and more zeros in the radial direction.

There is also appears to be an inertial wave $\omega = f$; in this limit $K \to 0$ and the solution to the interior equation is just $\phi = r^m$; this is consistent with the boundary condition $\phi_r(a) = m\phi(a)/a$. However, you cannot generally find a consistent radial flow structure that satisfies the boundary conditions; this is a spurious mode.

Forced Modes: in the simple case where the basin is small enough, we can take $\phi_e = \phi_0 x \cos(\omega t) = \phi_0 r \cos(\theta)\cos(\omega t)$; this has $\nabla^2\phi_e = 0$ so that the interior equation is still

$$\nabla^2\phi = \alpha^2(1-\omega^2)\phi$$

However ω is set by the tidal forcing, as is $m = 1$, so we will have I_1 solutions for sub-inertial frequencies and J_1 for super-inertial cases. The boundary conditions now dermine the amplitude of the waves. Since

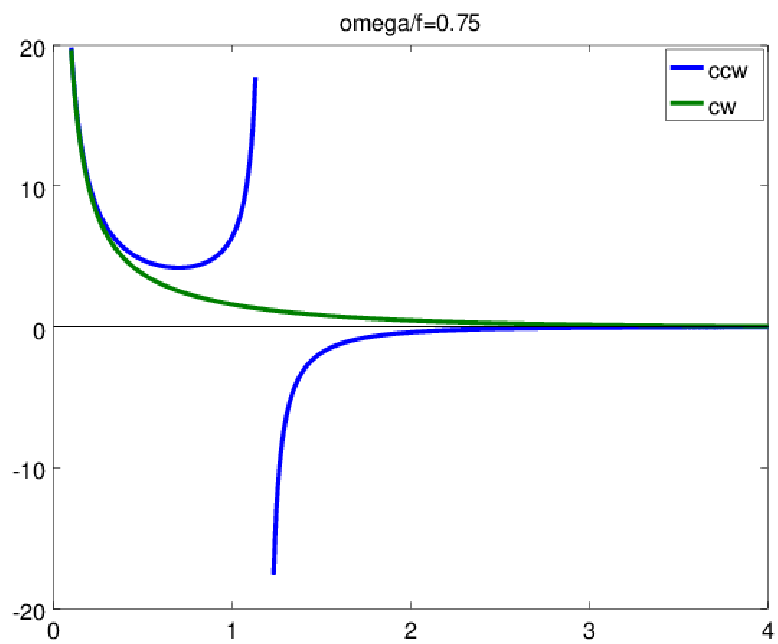
$$\cos(\theta)\cos(\omega t) = \frac{1}{2}[\cos(\theta+\omega t) + \cos(\theta-\omega t)]$$

we are forcing waves in both directions around the boundary. In terms of $\Re\exp(\imath\theta - \imath\omega t)$ we have both positive and negative frequencies, forced with equal amplitudes.

For subinertial cases, $\phi = A\frac{\phi_0}{2}I_1(Kr)$, we find

$$A[I_1(K)\frac{1}{\omega} - KI_1'(K)] = \frac{1}{\omega} - 1$$

Here $K = \alpha\sqrt{1-\omega^2}$.



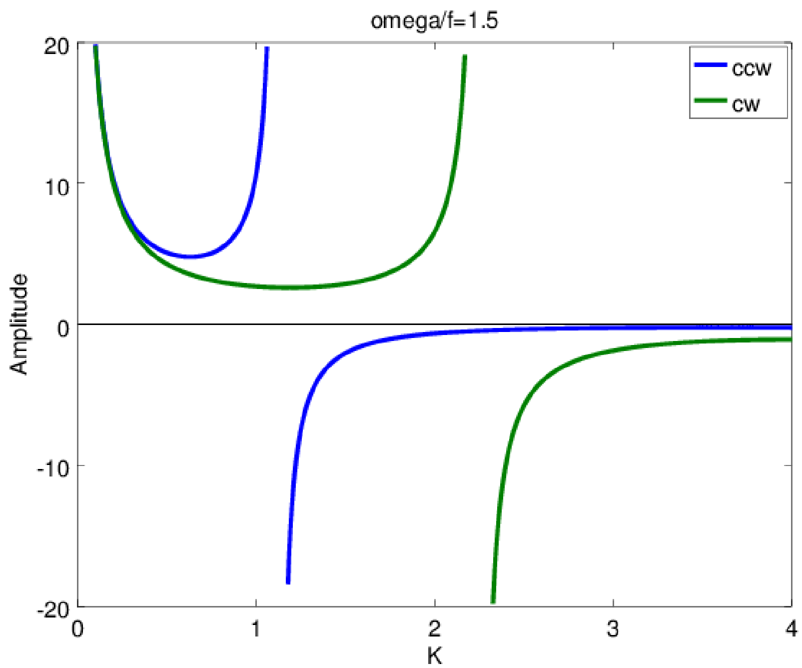
Amplitude as a function of basin size for subinertial motions $\omega = 0.75$

For the subinertial case, with I_1 solutions, the (single) singularity only occurs for the ccw ($n = 1$) solution since I_1' and I_1 are both positive.

The superinertial case has roots at all the free modes, both cw and ccw since J_1 and J_1' take on both signs..

$$A[J_1(K)\frac{1}{\omega} - KJ_1'(K)] = \frac{1}{\omega} - 1$$

with $K = \alpha\sqrt{\omega^2 - 1}$.



Amplitude as a function of basin size for superinertial motions $\omega/f = 1.5$

Note that the modes here are widely separated because the plots are for a small basin $\alpha = 1$.

Bowl

When $H = H(r)$, eqn (eq) is an ODE in r and should be solvable numerically. Free solutions should now include two gravity waves and a topographic Rossby wave; these should propagate ccw around the basin. So let's check that: suppose $H = H_0[1 - sr^2/2a^2] = H_0 h$ so that $H'/H_0 = -sr/a^2$, then

$$\nabla \cdot h\nabla\phi = \frac{f^2 - \omega^2}{gH_0}\phi - \frac{fns}{\omega a^2}\phi$$

$(h = 1 - s^2r^2/2a^2)$ with

$$\omega\frac{\partial}{\partial r}\phi = \frac{fn}{a}\phi$$

at $r = a$. We approximate $\omega << f$ and $s << 1$ to get

$$\nabla^2\phi = \frac{f^2}{gH_0}\phi - \frac{fns}{\omega a^2}\phi$$

with $\phi(a,\theta) = 0$. This has solutions

$$\phi = J_n(k_n r/a)$$

with k_n a zero of the Bessel function and

$$-k_n^2 - \frac{f^2}{gH_0} = -\frac{fns}{\omega a^2}$$

for $\omega > 0$ we have the $n > 0$ mode (ccw) with the shallow water on the right. Note that, for the forced problem, the solutions for $n = \pm 1$ will no longer be the same.