Shallow-water or long waves

For surface gravity waves, we can simplify the equations for the case of long waves (or shallow-water waves) from either the potential or the original momentum equations. We'll do the latter, starting with the Boussinesq equations

Digression – Boussinesq equations

$$\frac{\partial}{\partial t}\mathbf{u} + (\boldsymbol{\zeta} + \mathbf{f}) \times \mathbf{u} = -\frac{1}{\rho}\nabla p - \nabla\frac{1}{2}|\mathbf{u}|^2 - \nabla\Phi$$

$$\frac{D}{Dt}\ln\rho + \nabla\cdot\mathbf{u} = 0$$

$$\frac{D}{Dt}\rho - \frac{1}{c_s^2}\frac{D}{Dt}p = 0$$

with $c_s(p, \rho, S)$ the speed of sound. Let

$$\rho = \frac{\rho_0}{1+\tau} \quad , \quad p = \rho_0\tilde{P} - \rho_0\Phi$$

so that the r.h.s of the momentum equations becomes

$$-(1+\tau)\nabla\tilde{P} - \nabla\frac{1}{2}|\mathbf{u}|^2 + \tau\nabla\Phi$$

We'll take $|\tau| << 1$, giving

$$\frac{\partial}{\partial t}\mathbf{u} + (\boldsymbol{\zeta} + \mathbf{f}) \times \mathbf{u} = -\nabla\left[\tilde{P} + \frac{1}{2}|\mathbf{u}|^2\right] + \tau\nabla\Phi$$

and the mass equation

$$\frac{D}{Dt}\ln(1+\tau) = \nabla\cdot\mathbf{u} \quad \Rightarrow \quad \nabla\cdot\mathbf{u} = 0$$

Finally, the lowest order thermodynamic equation

$$-\rho_0\frac{D}{Dt}\tau - \frac{1}{c_{s0}^2}(-\rho_0\frac{D}{Dt}\Phi) = 0$$

(with c_{s0} the large constant value) gives the conservation of potential buoyancy

$$\frac{D}{Dt}b = 0 \quad , \quad b = g\tau - \frac{g}{c_{s0}^2}z$$

(with $\Phi = gz$). Finally, rewriting

$$\tilde{P} = P + \frac{1}{2}\frac{\Phi^2}{c_{s0}^2}$$

gives

$$\frac{\partial}{\partial t}\mathbf{u} + (\boldsymbol{\zeta} + \mathbf{f}) \times \mathbf{u} = -\nabla\left[P + \frac{1}{2}|\mathbf{u}|^2\right] + b\,\hat{\mathbf{z}}$$

$$\nabla\cdot\mathbf{u} = 0$$

$$\frac{D}{Dt}b = 0$$

From basic equations

For rotating, stratified flow with $\mathbf{f} = f\hat{\mathbf{z}}$, the vorticity evolves according to

$$\frac{\partial}{\partial t}\boldsymbol{\zeta} + \nabla \times ([\boldsymbol{\zeta} + f] \times \mathbf{u}) = \nabla \times (b\hat{\mathbf{z}}) = \nabla b \times \hat{\mathbf{z}}$$

If we dot this with a unit vector $\hat{\mathbf{n}}$, we find the evolution equation for the components of the absolute vorticity $\mathbf{Z} = \boldsymbol{\zeta} + \mathbf{f}$

$$\frac{\partial}{\partial t}Z_n + \nabla \cdot (\mathbf{u}Z_n) = \nabla \cdot (\mathbf{Z}u_n) + \hat{\mathbf{n}} \cdot (\nabla b \times \hat{\mathbf{z}})$$

or

$$\frac{\partial}{\partial t}Z_n + \mathbf{u} \cdot \nabla Z_n = \mathbf{Z} \cdot \nabla u_n + \nabla b \cdot (\hat{\mathbf{z}} \times \hat{\mathbf{n}})$$

Flow can remain irrotational when $\mathbf{f}$ and b are zero, but otherwise we generate vorticity by vortex stretching, tilting, and baroclinicity.

Hydrostatic

If $L >> H$, then the continuity equation implies $w \sim \frac{H}{L}\mathbf{u}_h = \delta \mathbf{u}_h$ and the w terms in the x and y components of $\boldsymbol{\zeta}$ are order δ^2 compared to the others:

$$\zeta_1 = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \simeq -\frac{\partial v}{\partial z}$$

so that the vorticity in the momentum equations is replaced by $\boldsymbol{\zeta}_h = \nabla \times \mathbf{u}_h$. Likewise the w^2 term in the Bernoulli function is order δ^2 compared to the others. Finally, if $P \sim UL/T$ then

$$\frac{[\frac{\partial}{\partial t}w]}{[\frac{\partial}{\partial z}P]} \sim \frac{UH/LT}{UL/HT} = \delta^2$$

Dropping all the δ^2 terms gives

$$\frac{\partial}{\partial t}\mathbf{u}_h + (\boldsymbol{\zeta}_h + f\hat{\mathbf{z}}) \times \mathbf{u} = -\nabla(P + \frac{1}{2}|\mathbf{u}_h|^2) + b\hat{\mathbf{z}}$$

Note that vertical advection is still significant:

$$[w\frac{\partial}{\partial z}] = U\frac{H}{L}\frac{1}{H} \sim [\mathbf{u}_h\frac{\partial}{\partial x}]$$

In conventional form, we have

$$\frac{D}{Dt}\mathbf{u}_h + f\hat{\mathbf{z}} \times \mathbf{u}_h = -\nabla_h P \quad , \quad \frac{\partial}{\partial z}P = b$$

The horizontal vorticity (e.g. Z_1) equation

$$\begin{aligned}\frac{D}{Dt}Z_1 &= (f + \zeta_3)\frac{\partial u}{\partial z} + +\zeta_1\frac{\partial}{\partial x}u + \zeta_2\frac{\partial}{\partial y}u + \hat{\mathbf{y}} \cdot \nabla b \\ &= (f + \zeta_3)\zeta_2 + \zeta_1\frac{\partial}{\partial x}u + \zeta_2\frac{\partial}{\partial y}u + \hat{\mathbf{y}} \cdot \nabla b\end{aligned}$$

shows that in the absence of buoyancy gradients, flow with $\zeta_1, \ \zeta_2$ initially zero will continue to have velocities without vertical shear.

Homogeneous fluid

Thus we have

$$\frac{\partial}{\partial z}\mathbf{u}_h = 0$$

The vertical momentum equation implies $\frac{\partial}{\partial z}P = 0$ and the continuity equation tells us that $\frac{\partial}{\partial z}w$ is independent of depth so that

$$\frac{\partial}{\partial z}w = \frac{w(\eta(\mathbf{x},t)) - w(-H(\mathbf{x},t))}{H(\mathbf{x},t)+\eta(\mathbf{x},t)} = \frac{1}{H+\eta}(\frac{\partial}{\partial t} + \mathbf{u}_h \cdot \nabla)(H+\eta)$$

Finally, we note that the pressure at the surface is

$$-\rho_0 g\eta(x,y,t) + \rho_0 P(x,y,t) = p_a(x,y,t)$$

where p_a is the atmospheric pressure. Thus

$$P = g\eta + \frac{1}{\rho_0}p_a$$

and our equations become

$$\frac{\partial}{\partial t}\mathbf{u}_h + (\zeta_3 + f)\hat{\mathbf{z}} \times \mathbf{u}_h + \nabla(\frac{1}{2}|\mathbf{u}_h|^2) = \frac{D}{Dt}\mathbf{u}_h + f\hat{\mathbf{z}} \times \mathbf{u}_h = -\nabla g\eta - \nabla\frac{p_a}{\rho_0}$$

$$\frac{\partial}{\partial t}(H+\eta) + \nabla \cdot [\mathbf{u}_h(H+\eta)] = 0$$

These are the "shallow water equations"

Irrotational case

When $f = 0$, ζ_3 will also stay zero, and we can use

$$\mathbf{u}_h = -\nabla\Phi$$

and the momentum equations give

$$\frac{\partial}{\partial t}\nabla\Phi = \nabla(g\eta + \frac{p_a}{\rho_0} + \frac{1}{2}|\nabla\Phi|^2) \quad \textit{or} \quad \frac{\partial}{\partial t}\Phi = g\eta + \frac{1}{2}|\nabla\Phi|^2 + \frac{p_a}{\rho_0}$$

and

$$\frac{\partial}{\partial t}(H+\eta) - \nabla \cdot [(H+\eta)\nabla\Phi = 0$$

(see below).

Linearized

To consider waves, we will linearize these equations

$$\boxed{\begin{aligned}&\frac{\partial}{\partial t}\mathbf{u} + f\hat{\mathbf{z}} \times \mathbf{u} = -\nabla g\eta - \nabla p_a/\rho_0 \\ &\frac{\partial}{\partial t}\eta + \nabla \cdot H\mathbf{u} = 0\end{aligned}}$$

In the absence of forcing, rotation, and topography

$$\frac{\partial^2}{\partial t^2}\eta = -H\nabla \cdot \frac{\partial \mathbf{u}}{\partial t} = gH\nabla^2\eta$$

– the ordinary wave equation (but 2D) – so that the wave speed is $\sqrt{gH}$. If we have topography, the same procedure gives

$$\frac{\partial^2}{\partial t^2}\eta = \nabla \cdot gH\nabla\eta$$

Potential

Our basic nonlinear equations in the case where the bottom depth varies $H = H_0 + h(\mathbf{x}, t)$ become

$$\begin{aligned}\nabla^2\phi &= 0 \\ \frac{\partial h}{\partial t} - \nabla\phi \cdot \nabla h &= \phi_z \quad at \quad z = -H_0 - h(\mathbf{x}, t) \\ \frac{\partial \eta}{\partial t} - \nabla\phi \cdot \nabla\eta &= -\phi_z \quad at \quad z = \eta(\mathbf{x}, t) \\ \frac{\partial \phi}{\partial t} &= g\eta + \frac{1}{2}|\nabla\phi|^2 \quad at \quad z = \eta\end{aligned}$$

If we nondimensionalize z by H_0, x, y by L, η by η_0, t by $L/\sqrt{gH_0}$, h by h_0 and ϕ by $g\eta_0 L/\sqrt{gH_0}$, we get

$$\begin{aligned}\frac{\partial^2\phi}{\partial z^2} + \delta^2\nabla_h^2\phi &= 0 \\ \epsilon_h\delta^2\frac{\partial h}{\partial t} - \epsilon_h\epsilon\delta^2\nabla\phi \cdot \nabla h &= \epsilon\phi_z \quad at \quad z = -1 + \epsilon_h h(\mathbf{x}, t) \\ \delta^2\frac{\partial \eta}{\partial t} - \delta^2\epsilon\nabla\phi \cdot \nabla\eta &= -\phi_z \quad at \quad z = \epsilon\eta(\mathbf{x}, t) \\ \frac{\partial \phi}{\partial t} &= \eta + \frac{\epsilon}{\delta^2}\frac{1}{2}\left(\frac{\partial \phi}{\partial z}\right)^2 + \epsilon|\nabla_h\phi|^2 \quad at \quad z = \epsilon\eta\end{aligned}$$

with $\delta = H_0/L$, $\epsilon = \eta_0/H_0$, and $\epsilon_h = h_0/H_0$. For the long-wave limit, we take $\delta^2 << 1$ and $\epsilon, \epsilon_h \sim 1$ (at least by comparison). Then the lowest order equations tell us that

$$\frac{\partial^2 \phi_0}{\partial z^2} = 0 \quad , \quad \frac{\partial \phi}{\partial z} = 0 \quad at \quad z = -1 + \epsilon_h h \quad , \quad \epsilon\eta$$

for which the solution is $\phi_0 = \Phi(x, y, t)$. This is consistent with the dynamic equation also. At the next order (δ^2), we find

$$\begin{aligned} \frac{\partial^2 \phi_1}{\partial z^2} &= -\nabla_h^2 \Phi \\ \epsilon_h \frac{\partial h}{\partial t} - \epsilon_h \epsilon \nabla\Phi \cdot \nabla h &= \epsilon \frac{\partial}{\partial z}\phi_1 \quad at \quad z = -1 + \epsilon_h h(\mathbf{x}, t) \\ \frac{\partial \eta}{\partial t} - \epsilon \nabla\Phi \cdot \nabla\eta &= -\frac{\partial}{\partial z}\phi_1 \quad at \quad z = \epsilon\eta(\mathbf{x}, t) \\ \frac{\partial \Phi}{\partial t} &= \eta + \epsilon|\nabla_h \Phi|^2 \quad at \quad z = \epsilon\eta \end{aligned}$$

Integrating Poisson's equation in z and applying the boundary conditions gives the mass conservation equation

$$\left(\frac{\partial}{\partial t} - \epsilon\nabla\Phi\right)\tilde{H} = \epsilon\tilde{H}\nabla_h^2\Phi$$

with the nondimensional depth of the fluid being $\tilde{H} = 1 + \epsilon\eta + \epsilon_h h$. The dynamic equation is

$$\frac{\partial}{\partial t}\Phi = \eta + \epsilon|\nabla_h \Phi|^2$$

If we look at linear, flat-bottom waves $h = 0$, $\epsilon << 1$ (but now requiring $\delta^2 << \epsilon << 1$), we have

$$\begin{aligned} \frac{\partial}{\partial t}\eta &= \nabla_h^2 \Phi \\ \frac{\partial}{\partial t}\Phi &= \eta \end{aligned}$$

giving the nondimensional wave equation

$$\frac{\partial^2}{\partial t^2}\eta = \nabla_h^2 \eta$$