Stokes waves (surface)

We start with the homogeneous fluid eqns.

$$\frac{\partial}{\partial t}\mathbf{u} + \boldsymbol{\zeta} \times \mathbf{u} = -\nabla(P + \frac{1}{2}\mathbf{u}\cdot\mathbf{u})$$
$$\nabla \cdot \mathbf{u} = 0$$

with pressure $p = -\rho_0 g z + \rho_0 P(x, y, z, t)$. For irrotational flow, $\boldsymbol{\zeta} = 0$ and

$$\mathbf{u} = -\nabla\phi$$

The dynamic equation is

$$\frac{\partial}{\partial t}\phi = P + \frac{1}{2}|\nabla\phi|^2$$

and

$$\nabla^2\phi = 0 \tag{1}$$

At the surface $p = 0 \;\rightarrow\; P = g\eta$ so that

$$\frac{\partial}{\partial t}\phi = g\eta + \frac{1}{2}|\nabla\phi|^2 \quad @\ z = \eta$$

and the kinematic condition becomes

$$\frac{\partial}{\partial t}\eta = \frac{\partial\phi}{\partial x}\frac{\partial\eta}{\partial x} - \frac{\partial\phi}{\partial z} \quad @\ z = \eta$$

We'll consider just deep water so that $\nabla\phi \rightarrow 0$ as $z \rightarrow -\infty$.

Non-dimensionalize: $t \sim 1/\sqrt{gk}$, $x \,\&\, z \sim 1/k$, $\eta \sim h$, $\phi \sim h\sqrt{g/k}$. Then the boundary conditions are

$$\frac{\partial}{\partial t}\eta = \epsilon\frac{\partial\phi}{\partial x}\frac{\partial\eta}{\partial x} - \frac{\partial\phi}{\partial z}$$
$$\frac{\partial}{\partial t}\phi = \eta + \epsilon\frac{1}{2}|\nabla\phi|^2$$

at $z = \epsilon\eta$ with $\epsilon = hk$ the wave steepness. We still have

$$\nabla^2\phi = 0$$

Expansion

$$\frac{\partial}{\partial t}\eta_0 = -\frac{\partial \phi_0}{\partial z}$$

$$\frac{\partial}{\partial t}\phi_0 = \eta_0$$

at $z = 0$.

$$\frac{\partial}{\partial t}\eta_1 = \frac{\partial \phi_0}{\partial x}\frac{\partial \eta_0}{\partial x} - \frac{\partial \phi_1}{\partial z} - \eta_0 \frac{\partial^2 \phi_0}{\partial z^2}$$

$$\frac{\partial}{\partial t}\phi_1 + \eta_0 \frac{\partial^2 \phi_0}{\partial t \partial z} = \eta_1 + \frac{1}{2}|\nabla \phi_0|^2$$

still at $z = 0$.

Lowest order solutions are

$$\eta_0 = \cos(x-t) \quad , \quad \phi_0 = -\sin(x-t)\exp(z)$$

Forcing terms for next order

$$\frac{\partial}{\partial t}\eta_1 = -\frac{\partial \phi_1}{\partial z} + \sin(2[x-t])$$

$$\frac{\partial}{\partial t}\phi_1 = \eta_1 - \frac{1}{2}\cos(2[x-t])$$

Using $\eta_1 = n_1 \cos(2[x-t])$, $\phi_1 = p_1 \sin(2[x-t])\exp(2z)$ gives

$$n_1 + p_1 = 1/2 \quad , \quad n_1 + 2p_1 = 1/2 \quad \Rightarrow \quad n_1 = 1/2 \quad , \quad p_1 = 0$$

Thus

$$\eta = \cos(x-t) + \frac{1}{2}\epsilon \cos(2[x-t])$$

or, dimensionally,

$$\eta = h\cos(k[x-ct]) + \frac{1}{2}kh^2 \cos(2k[x-ct])$$

with $c = \sqrt{g/k}$ from the space and time non-dimensionalization. (Note – the sign is different because η is the upward displacement [positive z] while Stokes dealt with a downward displacement [positive y in his paper].)