Surface gravity waves

For a homogeneous layer of fluid with a free surface at $z = \eta(x, y, t)$ and flat bottom at $z = -H$, we rewrite the pressure as $p = -\rho_0 g z + \rho_0 P(x, y, z, t)$ and the equations of motion become

$$\frac{\partial}{\partial t}\mathbf{u} + \boldsymbol{\zeta} \times \mathbf{u} = -\nabla(P + \frac{1}{2}\mathbf{u}\cdot\mathbf{u})$$
$$\nabla \cdot \mathbf{u} = 0$$

Irrotational flow: When the vorticity $\boldsymbol{\zeta} = \nabla \times \mathbf{u}$ is initially zero, it remains zero. We can see this from the vorticity equation derived by taking the curl of the momentum equation

$$\frac{\partial}{\partial t}\boldsymbol{\zeta} + \nabla \times (\boldsymbol{\zeta} \times \mathbf{u}) = 0$$

Therefore, if $\boldsymbol{\zeta}(\mathbf{x}, 0) = 0$, the vorticity will remain zero thereafter. In that case, the velocity is given by the gradient of a potential function

$$\mathbf{u} = -\nabla\phi$$

We can define a scalar function

$$\phi(\mathbf{x}_0) - \phi(\mathbf{x}) = \int_{\mathbf{x}_0}^{\mathbf{x}} \mathbf{u} \cdot d\ell$$

with the integral taken along a path joining the two points. The integral is path-independent, since the integral along a closed path $\oint \mathbf{u} \cdot d\ell = 0$ by Stokes' theorem. Thus ϕ with this definition is indeed a scalar function, and its derivatives with respect to $\mathbf{x}$ give the velocity components.

The equations in the interior of the fluid simplify to

$$-\nabla\frac{\partial}{\partial t}\phi = -\nabla(P + \frac{1}{2}|\nabla\phi|^2)$$
$$or$$
$$\frac{\partial}{\partial t}\phi = P + \frac{1}{2}|\nabla\phi|^2$$
$$\nabla^2\phi = 0$$

The first equation tells us how the pressure varies given the potential; it's the second equation that determines the structure of the field. The wave part of the dynamics doesn't appear obvious in Laplace's equation; instead it shows up in the boundary conditions.

Boundary conditions: The lower boundary condition is straightforward: it states that the normal component of velocity vanishes

$$w = -\frac{\partial \phi}{\partial z} = 0 \quad at \quad z = -H$$

Note that this has a broader implication. It implies that a particle of fluid on the bottom can move along the bottom but not off of it. In molecular terms, the condition simply states that the mean upward and downward velocities are equal. The fact that molecules can migrate away from the surface (and tracer material as well) is connected to the diffusive part of the dynamics, not the molecular mean velocity $\mathbf{u}$.

At the top surface, $z = \eta(x, y, t)$, we apply the same argument that particles remain at the interface; however, we must now account for the motion of the interface as well. We find

$$\frac{\partial}{\partial t}\eta + \mathbf{u}(x, y, \eta, t) \cdot \nabla\eta = w(x, y, \eta, t)$$

or

$$\frac{D}{Dt}(\eta - z) = 0 \quad with \quad \frac{D}{Dt} = \frac{\partial}{\partial t} - \nabla\phi \cdot \nabla$$

Finally, we apply a dynamic condition at the surface that the pressure of the fluid must equal the pressure of the air above

$$p_a = -\rho_0 g\eta + \rho_0 P(x, y, \eta, t) \quad \Rightarrow \quad P(x, y, \eta, t) = g\eta(x, y, t) + p_a/\rho_0$$

Using the Bernouilli equation allows us to write this condition in terms of the potential and the surface elevation

$$\frac{\partial}{\partial t}\phi - \frac{1}{2}|\nabla\phi|^2 = g\eta + p_a/\rho_0 \quad at \quad z = \eta$$

We could combine the kinematic and dynamic conditions to express the upper boundary condition solely in terms of ϕ; however, we will leave the two fields explicitly.

Linearized eqns: We can linearize the equations by assuming that velocities are small compared to the phase speed and changes in elevation are small compared tot he wavelength or the depth of the fluid. Linearization alters the upper boundary conditions in both the obvious way — dropping the $|\nabla\phi|^2$ and $\nabla\phi \cdot \nabla\eta$ terms — and by allowing the fields to be evaluated at $z = 0$ rather than $z = \eta$. Since $\phi_z(x, y, \eta, t) \simeq \phi_z(x, y, 0, t) + \eta\phi_z z(x, y, 0, t) + ...$, the correction terms from evaluating at η rather than 0 are indeed quadratic or higher order in the strength of the fields.

With these approximations, the equations become

$$\begin{aligned}
&\nabla^2\phi = 0 \\
&\frac{\partial}{\partial z}\phi = 0 \quad at \quad z = -H \\
&\frac{\partial}{\partial t}\phi = g\eta + p_a/\rho_0 \\
&\frac{\partial}{\partial t}\eta = -\frac{\partial}{\partial z}\phi \quad at \quad z = 0
\end{aligned}$$

DISPERSION RELATION: For $p_a = 0$ and solutions which are plane waves in the horizontal, we have

$$\eta = \eta_0 \exp(\imath \mathbf{k}\cdot\mathbf{x} - \imath\omega t)$$

so that

$$\phi = \imath \frac{g\eta_0}{\omega}\eta_0 e^{\imath\theta} \quad at` \quad z = 0$$

which, together with the lower bc. and Laplace's eqn., implies

$$\phi = \imath \frac{g}{\omega}\eta_0 \frac{\cosh(K[z+H])}{\cosh KH} e^{\imath\theta}$$

with $K \equiv |\mathbf{k}|$. The kinematic equation now tells us

$$\frac{\partial}{\partial t}\eta = -\imath \frac{gK}{\omega}\eta_0 \frac{\sinh(K[z+H])}{\cosh KH} e^{\imath\theta}$$

giving the dispersion relationship

$$\omega^2 = gK \tanh(KH)$$

Nondimensionally (using the scales H for length and $\sqrt{H/g}$ for time) we have

$$\omega^2 H//g = KH \tanh(KH) \quad or \quad \omega' = \pm \left[K' \tanh(K')\right]^{1/2}$$

Graphics Dispersion relation: `omega' vs K' log`
Short wave limit: For short waves, $KH >> 1 \Rightarrow \tanh(KH) = 1$ and

$$\omega \sim \sqrt{gK} \quad , \quad c = \sqrt{\frac{g}{K}} \quad , \quad \mathbf{c}_g = \frac{1}{2}\sqrt{\frac{g}{K}}\frac{\mathbf{k}}{K}$$

The group velocity is half the phase speed.
Long wave limit: For long waves, $KH << 1 \Rightarrow \tanh(KH) = KH$ and

$$\omega \sim \sqrt{gH}K \quad , \quad c = \sqrt{gH} \quad , \quad \mathbf{c}_g = \sqrt{gH}\frac{\mathbf{k}}{K}$$

The group velocity is equal to the phase speed.

Evolution of an initial disturbance

We shall now look at the evolution of an initial compact disturbance

$$\eta(\mathbf{x}, 0)$$

To see exactly what we need to specify, let's reformulate the equations a bit. From the momentum equations

$$\frac{\partial}{\partial t}\mathbf{u} = -\nabla P$$

and the continuity equation

$$\nabla \cdot \mathbf{u} = 0$$

we can see that the pressure also satisfies Laplace's equation

$$\nabla^2 P = 0$$

with an upper boundary condition

$$P = g\eta \quad at \quad z = 0$$

The lower boundary condition arises from

$$0 = \frac{\partial}{\partial t} w = -\frac{\partial}{\partial z} P \quad \Rightarrow \frac{\partial}{\partial z} P = 0 \quad at \quad z = -H$$

The kinematic condition at the upper boundary becomes

$$\frac{\partial}{\partial t}\eta = w \quad \Rightarrow \quad \frac{\partial^2}{\partial t^2}\eta = \frac{\partial}{\partial t} w = -\frac{\partial}{\partial z} P \quad at \quad z = 0$$

If we Fourier-analyze the surface elevation

$$\eta(x, t) = \iint d\mathbf{k}\, \hat{\eta}(\mathbf{k}, t) \exp(\imath \mathbf{k} \cdot \mathbf{x})$$

and the pressure

$$P(\mathbf{x}, z, t) = \iint d\mathbf{k}\, \hat{P}(\mathbf{k}, z, t) \exp(\imath \mathbf{k} \cdot x)$$

with $K = |\mathbf{k}|$, we have

$$\hat{P}(\mathbf{k}, 0, t) g\hat{\eta}(\mathbf{k}, t)$$

and, from the interior equation

$$\nabla^2 P = 0 = \iint d\mathbf{k} \left(\frac{\partial^2}{\partial z^2} - K^2 \right) \hat{P}(\mathbf{k}, z, t) \exp(\imath \mathbf{k} \cdot \mathbf{x})$$

Applying the lower boundary condition gives

$$\hat{P}(\mathbf{k}, x, t) = g\hat{\eta}(\mathbf{k}.t)\frac{\cosh K(z+H)}{\cosh KH}$$

and the kinematic condition gives

$$\frac{\partial^2}{\partial t^2}\hat{\eta} = -gK\tanh(KH)\hat{\eta} = -[\Omega(\mathbf{k})]^2\hat{\eta}$$

This equation makes it clear that we need two conditions at $t = 0$, one on η itself and one on $w = \frac{\partial}{\partial t}\eta$. Given these, we can write the general solution

$$\eta(\mathbf{x}, t) = \iint d\mathbf{k}\, \hat{\eta}_+(\mathbf{k})e^{\imath\mathbf{k}\cdot\mathbf{x} - \imath\Omega(\mathbf{k})t} + \hat{\eta}_-(\mathbf{k})e^{\imath\mathbf{k}\cdot\mathbf{x} + \imath\Omega(\mathbf{k})t}$$

with the first representing waves propagating in the positive $\mathbf{k}$ direction and the second representing waves moving in the opposite direction. These are related to the initial conditions by

$$\hat{\eta}(\mathbf{k}, 0) = \hat{\eta}_+ + \hat{\eta}_- \quad , \quad \frac{\partial}{\partial t}\hat{\eta}(\mathbf{k}, 0) = -\imath\Omega(\hat{\eta}_+ - \hat{\eta}_-)$$

Note: we can also write down the radially symmetric solutions; in the case with zero initial vertical velocity, we have

$$\eta(r, t) = \int_0^\infty kdk\, a(k)J_0(kr)\cos(\Omega(k)t)$$

One-D case: We shall look at the one-dimensional case for simplicity. Furthermore, we can look at only the part corresponding to eastward propagation. Thus, we seek an approximation to

$$\eta = \int dk\, \hat{\eta}_+(k)e^{\imath kx - \imath\Omega(k)t}$$

Consider large x and t but with ratio order 1. We can do this by setting $x = Ut$ and take the limit for large t. Then

$$\eta = \int dk\, \hat{\eta}_+(k)e^{\imath t\tilde{\theta}(k)} \quad , \quad \tilde{\theta}(k) = kU - \Omega(k)$$

The stationary phase method tells us that most of the contribution to the integral comes from the vicinity of k_s where $\tilde{\theta}'(k_s) = 0$. Elsewhere, the phase changes rapidly (for large t) and the integrand oscillates rapidly with zero net contribution. Alternatively we can move into the complex k plane and see that there is a saddle point in the phase at k_s; if we pass through this saddle point at a 45° angle, the argument of the exponential is strongly peaked. Therefore the main contribution comes from near the saddle point $c_g(k_s) = U$

$$\eta \sim \int dk\, \hat{\eta}_+(k_s) \exp\left(\imath t\tilde{\theta}(k_s) + \frac{1}{2}\imath t\tilde{\theta}''(k_s)(k-k_s)^2 + \dots\right)$$
$$\sim \hat{\eta}_+(k_s)e^{\imath t\tilde{\theta}(k_s)} \int dk \, \exp\left(\frac{1}{2}\imath t\tilde{\theta}''(k_s)(k-k_s)^2\right)$$

Treating the last integral as a probability integral with variance $\sqrt{-1/\imath t\tilde{\theta}''(k_s)}$ gives

$$\eta \sim \hat{\eta}_+(k_s)e^{\imath t\tilde{\theta}(k_s)+\imath\pi/4}\sqrt{\frac{2\pi}{\theta''(k_s)t}}$$

An observer moving at speed U sees waves with wavenumber k_s and frequency $\Omega(k_s)$ where $c_g(k_s) = U$; the wavenumber and frequency do not change for this observer; the amplitude does decrease as $t^{-1/2}$.

On the other hand, an observer at a fixed x corresponds to U decreasing with time and therefore k_s increasing; the wavelength and period get shorter and shorter as time increases. At fixed t, U increases with x, so that the longer waves appear at the front of the disturbance.

Since the details of the dispersion relation did not really enter, the result holds for any type of dispersive waves propagating in one direction; although we do need to watch out for issues such as the sign of θ'' etc. In two dimensions, the waves decay more rapidly $\sim t^{-1}$ (applying a similar asymptotic expansion to the Bessel function solution can show this).

Graphics Dispersion vs H: `H=20` `H=0.02` `max value`

Graphics Single pulse: `H=0.02` `H=0.2` `H=1` `H=2` `H=20`

Graphics Two-D: `top view h=10` `side view` `max value`