Ray Equations

In general, we deal with inhomogeneous media (e.g., variable depth), so that even the linear dynamics becomes more difficult to solve analytically. However, when the wavelengths are much smaller than the scale over which the medium changes, we can use WKB theory to make progress. Suppose the scale of the waves is ℓ and the scale for the medium changes is L with $\ell/L \equiv \epsilon$. Then we can look for solutions with

$$\psi = A(\mathbf{X}, T) \exp\left(\imath \frac{1}{\epsilon} \theta(\mathbf{X}, T)\right)$$

where $X = \epsilon \mathbf{x}$. Thus when $\mathbf{x}$ changes by order one, $\mathbf{X}$ changes by order ϵ and θ changes by order ϵ and the phase $\Theta = \theta/\epsilon$ changes by order 1. Spatial gradients will be computed using

$$\frac{\partial}{\partial x} = \epsilon \frac{\partial}{\partial X}$$

so that

$$\frac{\partial}{\partial X} \psi = \frac{\partial \theta}{\partial X} A e^{\imath \theta/\epsilon} + \epsilon \frac{\partial A}{\partial X} e^{\imath \theta/\epsilon}$$

If we make such substitutions for all the variables in the problem, the dynamics looks exactly like the linear dynamics locally with $\mathbf{k} \to \boldsymbol{\nabla} \theta$ and $\omega \to -\frac{\partial}{\partial T} \theta$. Note that we may need $T = \epsilon^n t$ rather than order $n = 1$ depending on the relationship between wavenumbers and frequencies. These identifications make sense: locally the phase looks like

$$\begin{aligned} \Theta &= \Theta_0 + \mathbf{x} \cdot \nabla \Theta + t \frac{\partial}{\partial t} \Theta \\ &= \Theta_0 + \mathbf{x} \cdot \boldsymbol{\nabla} \theta + t \frac{\partial}{\partial T} \theta \end{aligned}$$

Thus, the lowest order dynamics give

$$\boxed{\frac{\partial}{\partial T} \theta = -\Omega(\boldsymbol{\nabla} \theta | \mathbf{X}, T) \qquad (1)}$$

It may be possible to solve this directly (at least numerically).

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However, we can also look at how wavenumbers and frequencies change with time along a ray. Let's begin by taking a T derivative of (1)

$$\frac{\partial}{\partial T}\frac{\partial\theta}{\partial T} = -\boldsymbol{\nabla}_j\frac{\partial\theta}{\partial T}\cdot\frac{\partial}{\partial k_j}\Omega - \frac{\partial}{\partial T}\Omega$$

where the first term on the right represents time changes associated with the wavenumbers – the first set of arguments $\boldsymbol{\nabla}\theta$ of the dispersion relation function Ω – while the second represents changes arising from time variations in the media (e.g., large scale flow,...) – the second set of arguments in Ω. From the definition of group velocity, we have

$$\left(\frac{\partial}{\partial T} + \mathbf{c}_g\cdot\boldsymbol{\nabla}\right)\frac{\partial\theta}{\partial T} = -\frac{\partial}{\partial T}\Omega$$

or

$$\boxed{\left(\frac{\partial}{\partial T} + \mathbf{c}_g\cdot\boldsymbol{\nabla}\right)\omega = \frac{\partial}{\partial T}\Omega \qquad (2)}$$

Taking an X derivative of (1) gives

$$\frac{\partial}{\partial T}\frac{\partial\theta}{\partial X} = -\boldsymbol{\nabla}_j\frac{\partial\theta}{\partial X}\frac{\partial}{\partial k_j}\Omega - \frac{\partial}{\partial X}\Omega$$

or

$$\boxed{\left(\frac{\partial}{\partial T} + \mathbf{c}_g\cdot\boldsymbol{\nabla}\right)\mathbf{k} = -\boldsymbol{\nabla}\Omega \qquad (3)}$$

Equations (2) and (3) ensure that as we move along a ray with the "wave" at position given by

$$\dot{\mathbf{X}} = \mathbf{c}_g$$

the frequency and wavenumbers remain consistent with the local dispersion relation. To see this, let us check that

$$\omega(\mathbf{X} + \mathbf{c}_g\delta T, T + \delta T) = \Omega(\mathbf{k}(\mathbf{X} + \mathbf{c}_g\delta T, X + \mathbf{c}_g\delta T, T + \delta T)$$

This requires

$$\left(\frac{\partial}{\partial T} + \mathbf{c}_g\cdot\boldsymbol{\nabla}\right)\omega\delta T = \frac{\partial\Omega}{\partial k_j}\left(\frac{\partial}{\partial T} + \mathbf{c}_g\cdot\boldsymbol{\nabla}\right)k_j\delta T + \mathbf{c}_g\cdot\boldsymbol{\nabla}\Omega\delta T + \frac{\partial}{\partial T}\Omega\delta T$$

Using (2) and (3) (and cancelling the δT factor) gives

$$\frac{\partial}{\partial T}\Omega = -\frac{\partial\Omega}{\partial k_j}\frac{\partial\Omega}{\partial X_j} + \mathbf{c}_g\cdot\boldsymbol{\nabla}\Omega + \frac{\partial}{\partial T}\Omega$$

which is indeed true.