Solitary waves

We start with the x and z momentum equations and look at the sizes of the various terms using $x \sim L$. $z \sim H$, $t \sim L/c$, $u \sim U$, $w \sim UH/L$, $P \sim cU$

$$\begin{array}{ccccc} \dfrac{\partial}{\partial t}u = & -\dfrac{\partial}{\partial x}\Big[P & +\dfrac{1}{2}u^2 & +\dfrac{1}{2}w^2\Big] \\ Uc/L & Uc/L & U^2/L & U^2H^2/L^3 \\ 1 & 1 & \epsilon & \epsilon\delta^2 \end{array}$$

$$\begin{array}{ccccc} \dfrac{\partial}{\partial t}w = & -\dfrac{\partial}{\partial z}\Big[P & +\dfrac{1}{2}u^2 & +\dfrac{1}{2}w^2\Big] \\ UcH/L^2 & Uc/H & U^2/H & U^2H/L^2 \\ \delta^2 & 1 & \epsilon & \epsilon\delta^2 \end{array}$$

The parameters are $\epsilon = U/c$ and $\delta = H/L$. The second lines show the ratio of terms.

The two terms in the continuity equation are the same size.

$$\frac{\partial}{\partial x}u + \frac{\partial}{\partial z}w = 0$$

The lower boundary condition is

$$w(-H) = 0$$

and the upper boundary conditions become (with $\eta \sim UH/c$)

$$\begin{array}{cccccc} \dfrac{\partial}{\partial t}\eta & +u\dfrac{\partial}{\partial x}\eta & = w & @ & z = \eta \\ UH/L & U^2H/cL & UH/L & @ & H & UH/c \\ 1 & \epsilon & 1 & @ & 1 & \epsilon \end{array}$$

and

$$\begin{array}{cccccc} P = & g\eta & @ & z = \eta \\ Uc & gHU/c & @ & H & UH/c \\ 1 & gH/c^2 = 1 & @ & 1 & \epsilon \end{array}$$

We look in a frame moving with the long wave speed $c = \sqrt{gH}$, $u = u(x - ct, \epsilon t) = u_0 + \epsilon u_1$ etc. At order 1, we will assume $\epsilon \sim \delta^2$. The lowest order equations are

$$-c\frac{\partial}{\partial x}u_0 = -\frac{\partial}{\partial x}P_0$$

$$0 = -\frac{\partial}{\partial z}P_0$$

$$\frac{\partial}{\partial x}u_0 + \frac{\partial}{\partial z}w_0 = 0$$

$$w_0(-H) = 0$$

$$P_0(0) = g\eta_0$$

$$-c\frac{\partial}{\partial x}\eta_0 = w(0)$$

and give

$$P_0 = g\eta_0 \quad , \quad u_0 = g\eta_0/c \quad , \quad \frac{\partial}{\partial z} w_0 = -g/c \frac{\partial}{\partial x}\eta_0 = -c\frac{\partial}{\partial x}\eta/H$$

implying $c = \sqrt{gH}$ and

$$w_0 = -\frac{z+H}{H} c \frac{\partial}{\partial x}\eta$$

The first order equations are

$$\begin{aligned}
\frac{\partial}{\partial T}u_0 - c\frac{\partial}{\partial x}u_1 &= -\frac{\partial}{\partial x}P_1 - \frac{\partial}{\partial x}\frac{1}{2}u_0^2 \\
-c\frac{\partial}{\partial x}w_0 &= -\frac{\partial}{\partial z}P_1 \\
\frac{\partial}{\partial x}u_1 + \frac{\partial}{\partial z}w_1 &= 0 \\
w_1(-H) &= 0 \\
P_1 + \eta_0 \frac{\partial}{\partial z}P_0 &= g\eta_1 \quad @ \quad z = 0
\end{aligned}$$

or

$$P_1 = g\eta_1$$

$$\frac{\partial}{\partial T}\eta_0 - c\frac{\partial}{\partial x}\eta_1 + u_0\frac{\partial}{\partial x}\eta_0 = w_1 + \eta_0\frac{\partial}{\partial z}w_0 \quad @ \quad z = 0$$

Integrate

$$\frac{\partial}{\partial z}P_1 = c\frac{\partial}{\partial x}w_0 \quad \Rightarrow \quad P_1 = g\eta_1 - c^2(z + \frac{z^2}{2H})\eta_{xx}$$

(dropping the subscript on η_0) and put in the x-momentum equation

$$\frac{\partial}{\partial z}w_1 = -\frac{g}{c^2}\eta_T - \frac{g}{c}\frac{\partial}{\partial x}\eta_1 - c(z + \frac{z^2}{2H})\eta_{xxx} - \frac{g^2}{c^3}\eta\eta_x$$

Integrate again from $z = -H$ to $z = 0$ and apply the surface condition

$$w_1 = \eta_T - c\frac{\partial}{\partial x}\eta_1 + \frac{2g}{c}\eta\frac{\partial}{\partial x}\eta$$

Some algebra and use of $gH = c^2$ gives

$$\eta_T = -\frac{3g}{2c}\eta\eta_x - \frac{cH^2}{6}\eta_{xxx}$$

For steadily propagating disturbances $\eta_T = -c_1\eta_x$

$$c_1\eta = \frac{3g}{4c}\eta^2 + \frac{cH^2}{6}\eta_{xx}$$

For the linear waves $c_1 = -\frac{c}{6}k^2H^2$, which is the correct term in the expansion of $c = sqrtg \tanh(kH)/k$. This has periodic (cnoidal wave) solutions,

$$\eta = A\,\mathrm{sech}^2(kx) \quad \Rightarrow \quad c_1 = \frac{2}{3}ck^2H^2 \quad \text{and} \quad A = \frac{4}{3}k^2H^3$$

The phase speed correcti is **positive**: the waves travel faster than the longest linear waves (consistent with imaginary wavenumber in the exterior). The amplitude is always positive (so that the curvature can be negative). The speed depends on the amplitude as does the scale $1/k$.

Graphics Collision: `two waves` `summary`

Graphics kdv solns: `dispersive amp=0.1` `one soliton amp=1` `two amp=3` `three amp=7`