Inhomogeneous media - wave action

We shall consider long waves embedded in a medium with variable depth H and mean flows $\mathbf{u}$ (all vectors, gradient,... are horizontal for this section). The shallow water equations are

$$\frac{\partial}{\partial t}\phi = g\eta + \frac{1}{2}|\nabla\phi|^2$$
$$\frac{\partial}{\partial t}\eta = \nabla\cdot[(H+\eta)\nabla\phi]$$

We'll assume that the background is a large-scale flow $\overline{\mathbf{u}}(\mathbf{X},T)$ and depth field $H(\mathbf{X}) + \overline{\eta}(\mathbf{X},T)$ [which we call $\overline{H}(\mathbf{X},T)$]. These vary only on space and time scales which are long compared to the wave scales and periods ($\mathbf{X} = \epsilon\mathbf{x}$, $T = \epsilon t$). The perturbations satisfy

$$D\phi' = g\eta'$$
$$D\eta' + \eta'\nabla\cdot\overline{\mathbf{u}} = \nabla\cdot[\overline{H}\nabla\phi']$$

with $D \equiv \frac{\partial}{\partial t} - \nabla\overline{\phi}\cdot\nabla = \frac{\partial}{\partial t} + \overline{\mathbf{u}}\cdot\nabla$. If we now think of the waves as having structure

$$\eta' = \tilde{\eta}(\mathbf{X},T)\exp\left(\imath\frac{1}{\epsilon}\theta(\mathbf{X},T)\right)$$

(meaning the real part, of course), an operation such as $D\eta'$ becomes

$$\begin{aligned}D\eta' &= \epsilon\left(\frac{\partial}{\partial T} + \overline{u}\frac{\partial}{\partial X} + \overline{v}\frac{\partial}{\partial Y}\right)\eta' \\ &= e^{\imath\theta/\epsilon}\left[\imath\left(\frac{\partial\theta}{\partial T} + \overline{u}\frac{\partial\theta}{\partial X} + \overline{v}\frac{\partial\theta}{\partial Y}\right)\tilde{\eta} + \epsilon\left(\frac{\partial}{\partial T} + \overline{u}\frac{\partial}{\partial X} + \overline{v}\frac{\partial}{\partial Y}\right)\tilde{\eta}\right] \\ &= e^{\imath\theta/\epsilon}\left[\imath D_0\tilde{\eta} + \epsilon D_1\tilde{\eta}\right]\end{aligned}$$

where D_0 is the algebraic quantity

$$D_0 = \frac{\partial\theta}{\partial T} + \overline{u}\frac{\partial\theta}{\partial X} + \overline{v}\frac{\partial\theta}{\partial Y}$$

and D_1 is the operator

$$D_1 = \frac{\partial}{\partial T} + \overline{u}\frac{\partial}{\partial X} + \overline{v}\frac{\partial}{\partial Y}$$

Likewise the gradient operator will pick up two terms, one from the phase and one from the slow variations

$$\frac{\partial}{\partial x} \longrightarrow \imath\frac{\partial\theta}{\partial X} + \epsilon\frac{\partial}{\partial X} \quad , \qquad \nabla \longrightarrow \imath\mathbb{\nabla}\theta + \epsilon\mathbb{\nabla}$$

Our equations now become

$$\imath D_0\tilde{\phi} + \epsilon D_1\tilde{\phi} = g\tilde{\eta}$$
$$\imath D_0\tilde{\eta} + \epsilon D_1\tilde{\eta} + \epsilon\tilde{\eta}\boldsymbol{\nabla}\cdot\overline{\mathbf{u}} = (\imath\boldsymbol{\nabla}\theta + \epsilon\boldsymbol{\nabla})\cdot[\overline{H}(\imath\boldsymbol{\nabla}\theta + \epsilon\boldsymbol{\nabla})\tilde{\phi}]$$

and we can expand

$$\tilde{\eta} = \eta + \epsilon\eta_1 + \ldots \quad etc.$$

At lowest order, we get the local wave equations

$$\imath D_0\phi = g\eta$$
$$\imath D_0\eta = -\overline{H}|\boldsymbol{\nabla}\theta|^2\phi$$

which gives the dispersion relation

$$D_0^2 = (\omega - \overline{\mathbf{u}}\cdot\boldsymbol{\nabla}\theta)^2 = g\overline{H}|\boldsymbol{\nabla}\theta|^2 \equiv \hat{\omega}^2$$

Here $\hat{\omega}$ is the intrinsic frequency (that for waves in a medium at rest) and the frequency ω has both an advective and a wave contribution

$$\omega = \overline{\mathbf{u}}\cdot\boldsymbol{\nabla}\theta + \hat{\omega}$$

The amplitude is determined by the first order equations

$$\imath D_0\phi_1 + D_1\phi = g\eta_1$$
$$\imath D_0\eta_1 + D_1\eta + \eta\boldsymbol{\nabla}\cdot\overline{\mathbf{u}} = -H|\boldsymbol{\nabla}\theta|^2\phi_1 + \imath\overline{H}\boldsymbol{\nabla}\theta\cdot\boldsymbol{\nabla}\phi + \imath\boldsymbol{\nabla}\cdot[\overline{H}\phi\boldsymbol{\nabla}\theta]$$

Again, we multiply the first equation by $\imath D_0$ and the second by g and add. The η_1 and ϕ_1 terms cancel (using the dispersion relation) and we are left with

$$\imath D_0 D_1\phi + D_1 g\eta + g\eta\boldsymbol{\nabla}\cdot\overline{\mathbf{u}} = \imath g\overline{H}\boldsymbol{\nabla}\theta\cdot\boldsymbol{\nabla}\phi + \imath\boldsymbol{\nabla}\cdot[g\overline{H}\phi\boldsymbol{\nabla}\theta]$$

We substitute the lowest order expression $g\eta = \imath D_0\phi$ and divide by $\imath$ to get the equation for the evolution of the amplitude

$$D_0 D_1\phi + D_1(D_0\phi) + D_0\phi\boldsymbol{\nabla}\cdot\overline{\mathbf{u}} = g\overline{H}\boldsymbol{\nabla}\theta\cdot\boldsymbol{\nabla}\phi + \boldsymbol{\nabla}\cdot[g\overline{H}\phi\boldsymbol{\nabla}\theta]$$

or

$$2D_0 D_1\phi - 2g\overline{H}\boldsymbol{\nabla}\theta\cdot\boldsymbol{\nabla}\phi + \phi\left[D_1 D_0 + D_0\boldsymbol{\nabla}\cdot\overline{\mathbf{u}} - \boldsymbol{\nabla}\cdot(g\overline{H}\boldsymbol{\nabla}\theta)\right] = 0$$

To put this in terms of the energy,

$$E = \frac{1}{2}\overline{H}|\boldsymbol{\nabla}\theta|^2|\phi|^2 + \frac{1}{2}g|\eta|^2 = \frac{1}{g}\left[g\overline{H}|\boldsymbol{\nabla}\theta|^2|\phi|^2 + \frac{1}{2}D_0^2|\phi|^2\right] = \hat{\omega}^2|\phi|^2/g$$

we multiply the equation by $\frac{1}{2}\phi^*$ and add the conjugate to get

$$D_0 D_1 |\phi|^2 - g\overline{H}\boldsymbol{\nabla}\theta \cdot \boldsymbol{\nabla}|\phi|^2 + |\phi|^2 \left[D_1 D_0 + D_0 \boldsymbol{\nabla} \cdot \overline{\mathbf{u}} - \boldsymbol{\nabla} \cdot (g\overline{H}\boldsymbol{\nabla}\theta)\right] = 0$$

Combining the first and third terms and the second and fifth gives

$$D_1(\hat{\omega}|\phi|^2) + \nabla \cdot [g\overline{H}\boldsymbol{\nabla}\theta|\phi|^2] + |\phi|^2\hat{\omega}\boldsymbol{\nabla} \cdot \overline{\mathbf{u}} = 0$$

or

$$\frac{\partial}{\partial t}(\hat{\omega}|\phi|^2) + \nabla \cdot [(\overline{\mathbf{u}} + \frac{g\overline{H}}{\hat{\omega}}\boldsymbol{\nabla}\theta)\hat{\omega}|\phi|^2] = 0$$

But from the equation for the frequency, we find the group velocity is

$$\mathbf{c}_g = \overline{\mathbf{u}} + \sqrt{g\overline{H}}\frac{\nabla\theta}{|\nabla\theta|} = \overline{\mathbf{u}} + \frac{g\overline{H}}{\hat{\omega}}\boldsymbol{\nabla}\theta$$

and our equation for the so-called "wave action" $A = \hat{\omega}|\phi|^2/g = E/\hat{\omega}$ becomes

$$\frac{\partial}{\partial t}A + \boldsymbol{\nabla} \cdot (\mathbf{c}_g A) = 0$$

Action changes locally by fluxing in or out at the group velocity. Note that it is the energy divided by the intrinsic frequency which can now be balanced out, not the energy itself:

$$\frac{\partial}{\partial t}\frac{E}{\hat{\omega}} + \boldsymbol{\nabla} \cdot \left(\mathbf{c}_g \frac{E}{\hat{\omega}}\right) = 0$$

EXAMPLES: For a first example, consider waves traveling into shallow water, $H = H(x)$. We'll start with the waves impinging at an angle with wavenumber $\mathbf{k}_0$. The equations for changes along a ray

$$\begin{aligned}
\frac{\partial\omega}{\partial t} + \mathbf{c}_g \cdot \nabla\omega &= 0 \\
\frac{\partial k}{\partial t} + \mathbf{c}_g \cdot \nabla k &= -\frac{1}{2}\sqrt{\frac{g}{H}}\frac{\partial H}{\partial x}K \\
\frac{\partial \ell}{\partial t} + \mathbf{c}_g \cdot \nabla\ell &= 0
\end{aligned}$$

imply that the frequency and y-wavenumber remain fixed. Therefore, when the wave reached position X, its wavenumber is

$$K = \sqrt{\frac{H_0}{H(X)}}K_0 \quad , \quad k = \sqrt{\frac{H_0}{H(X)}K_0^2 - \ell_0^2}$$

The ray itself satisfies the equation

$$\frac{d\mathbf{X}}{dt} = \sqrt{gH(\mathbf{X})}\frac{\mathbf{k}(\mathbf{X})}{K(\mathbf{X})} = \frac{gH(\mathbf{X})}{\omega}\mathbf{k}(\mathbf{X})$$

from which we conclude that

$$\frac{dY}{dX} = \frac{\ell_0}{k(\mathbf{X})} = \left(\frac{H_0 K_0^2}{H(X)\ell_0^2} - 1\right)^{-1/2}$$

For $H(x) = H_0 \exp(-\gamma x)$, we have

$$Y = \frac{2}{\gamma}\tan^{-1}\left(\sqrt{\frac{K_0^2}{\ell_0^2}e^{\gamma x} - 1}\right)$$

But the essential character is clear from the expressions for the wavenumber and the trajectories: as the depth decreases, the cross-shelf wavenumber increases so that the waves are short and align more parallel with the coast. The trajectory slope decreases, again indicating a turning until the waves are propagating perpendicular to the coast.

Since the intrinsic frequency is just ω, and it's fixed, the energy satisfies

$$\frac{\partial}{\partial t}E + \mathbf{c}_g \cdot \nabla E = -E\nabla \cdot \mathbf{c}_g = -\frac{gE}{\omega}\frac{\partial}{\partial x}(Hk)$$

As the depth gets small, the group velocity behaves as $\sqrt{H}$ so that the fluxes are convergent and the energy density increases.

If we add an along-shore current $v(x)$, the dispersion relation becomes

$$\omega = v\ell + \sqrt{gH(x)}K$$

and ω and ℓ are still invariant along the trajectories. If $v(x)$ exceeds $\sqrt{gH_0}$, we will see reflection of some waves back off-shore. Otherwise if the along-shore velocity reaches some limit as the water shoals, K will still increase as $H^{-1/2}$ so the waves still become parallel to the shore. However, the intrinsic frequency is now $\hat{\omega} = \omega - v\ell$ and decreases as the velocity increases. Thus the increases in wave action associated with the convergence of $\mathbf{c}_g$ will not be entirely reflected in the energy $E = \hat{\omega}A$.

`Graphics, Page 4: Propagation onto shelf linear nonlinear 2d amplitude`