Group velocity

Phase and group velocities

The phase of the wave is $\theta = \mathbf{k}\cdot\mathbf{x} - \omega t$, and the rate of movement in the direction parallel to the wavenumber vector is

$$\delta\theta = \mathbf{k}\cdot\hat{\mathbf{k}}\,c\,\delta t - \omega\delta t = 0 \quad \Rightarrow \quad \delta t\Big[|\mathbf{k}|c - \omega\Big] = 0$$

so that

$$c = \frac{\omega}{|\mathbf{k}|}$$

But a packet of waves (and the energy) doesn't propagate like that at all. To see how it does, let's suppose the initial condition has a sharply-peaked spectrum

$$\hat{\eta}(\mathbf{k}, 0) = \epsilon^{-3}\phi(\frac{\mathbf{k}-\mathbf{k}_0}{\epsilon})$$

so that the initial condition represents a large-scale modulation of a small-scale wave

$$\begin{aligned}
\eta(\mathbf{x}, t) &= \int d^2\mathbf{k}\,\epsilon^{-3}\phi(\frac{\mathbf{k}-\mathbf{k}_0}{\epsilon})\exp(\imath\mathbf{k}\cdot\mathbf{x}) \\
&= \int d^2\mathbf{K}\,\phi(\mathbf{K})\exp(\imath\mathbf{k}_0\cdot\mathbf{x} + \imath\epsilon\mathbf{K}\cdot\mathbf{x}) \\
&= \exp(\imath\mathbf{k}_0\cdot\mathbf{x})\int d^3\mathbf{K}\,\phi(\mathbf{K})\exp(\imath\mathbf{K}\cdot\epsilon\mathbf{x}) \\
&= A(\epsilon\mathbf{x})\exp(\imath\mathbf{k}_0\cdot\mathbf{x})
\end{aligned}$$

`Graphics: Wave packet wave packet fft spectrum spectrum evolution c=0.58 cg=0.30`

The time-dependent solution is

$$\begin{aligned}
\eta(\mathbf{x}, t) &= \int d^2\mathbf{k}\,\epsilon^{-3}\phi(\frac{\mathbf{k}-\mathbf{k}_0}{\epsilon})\exp(\imath\mathbf{k}\cdot\mathbf{x} - \Omega(\mathbf{k})t) \\
&= \int d^2\mathbf{K}\,\phi(\mathbf{K})\exp(\imath\mathbf{k}_0\cdot\mathbf{x} + \imath\epsilon\mathbf{K}\cdot\mathbf{x} - \imath\Omega(\mathbf{k}_0 + \epsilon\mathbf{K})t) \\
&= \exp(\imath\mathbf{k}_0\cdot\mathbf{x} - \imath\Omega(\mathbf{k}_0)t)\int d^2\mathbf{K}\,\phi(\mathbf{K})\exp(\imath\mathbf{K}\cdot\epsilon\mathbf{x} - \imath\Omega(\mathbf{k}_0 + \epsilon\mathbf{K})t + \imath\Omega(\mathbf{k}_0)t) \\
&\simeq \exp(\imath\mathbf{k}_0\cdot\mathbf{x} - \imath\Omega(\mathbf{k}_0)t)\int d^2\mathbf{K}\,\phi(\mathbf{K})\exp(\imath\mathbf{K}\cdot\epsilon\mathbf{x} - \imath\mathbf{K}\cdot\nabla_{\mathbf{k}}\Omega(\mathbf{k}_0)\epsilon t) \\
&= A(\epsilon[\mathbf{x} - \nabla_{\mathbf{k}}\Omega\,t])\exp(\imath\mathbf{k}_0\cdot\mathbf{x} - \imath\Omega(\mathbf{k}_0)t)
\end{aligned}$$

Thus the envelope propagates at the group velocity

$$\mathbf{c}_g = \nabla_k\Omega|_{\mathbf{k}_0} \quad or \quad c_{g\,i} = \frac{\partial\Omega}{\partial k_i}$$

For surface waves,

$$\Omega^2 = g|\mathbf{k}|\tanh(|\mathbf{k}|H)$$

and

$$\nabla_k|\mathbf{k}| = \frac{\mathbf{k}}{|\mathbf{k}|}$$

so that

$$2\Omega\mathbf{c}_g = \mathbf{k}gH\left[\frac{\tanh(|\mathbf{k}|H)}{|\mathbf{k}|H} + \mathrm{sech}^2(|\mathbf{k}|H)\right]$$

Thus

$$\mathbf{c}_g = \frac{1}{2}\sqrt{gH}\frac{\mathbf{k}}{|\mathbf{k}|}\left[\sqrt{\frac{\tanh|\mathbf{k}|H}{|\mathbf{k}|H}} + \sqrt{\frac{|\mathbf{k}|H}{\tanh|\mathbf{k}|H}}\mathrm{sech}^2|\mathbf{k}|H\right]$$

which has the limits

$$\mathbf{c}_g \to \sqrt{gH}\frac{\mathbf{k}}{|\mathbf{k}|} \qquad |\mathbf{k}| \to \infty$$

and

$$\mathbf{c}_g \to \frac{1}{2}\sqrt{\frac{g}{|\mathbf{k}|}}\frac{\mathbf{k}}{|\mathbf{k}|} \qquad |\mathbf{k}| \to 0$$

Another view of group velocity

Consider superimposing two waves,

$$\eta = 0.5\cos(k_1x - \omega_1t) + 0.5\cos(k_2x - \omega_2t)$$

with $k_1 = k - \frac{\Delta k}{2} < k_2 = k + \frac{\Delta k}{2}$ and $\omega_1 = \omega - \frac{\Delta\omega}{2}$, $\omega_2 = \omega + \frac{\Delta\omega}{2}$; the result has a "beat-frequency" modulation.

$$\eta = \cos(kx - \omega t)\cos(\frac{\Delta k}{2}x - \frac{\Delta\omega}{2}t)$$

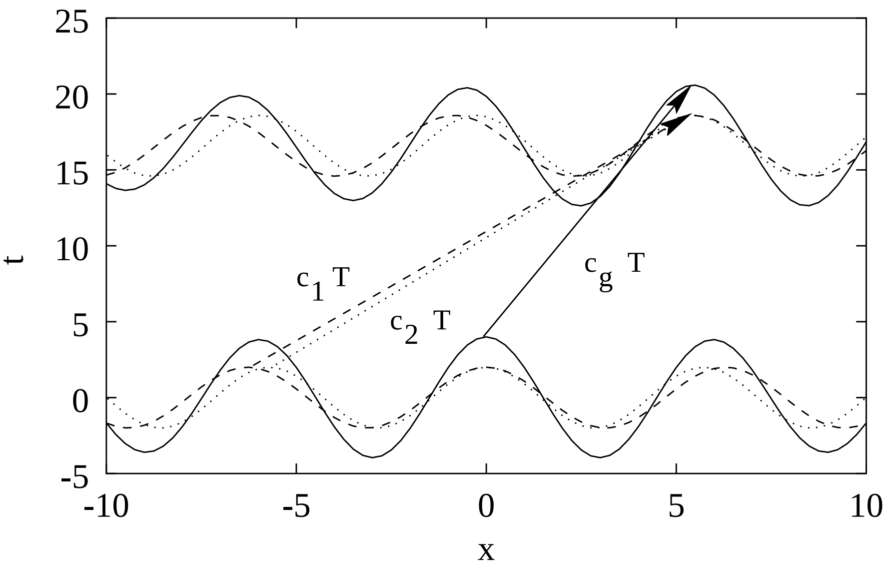
which propagates at a speed

$$c_g = \frac{\Delta\omega}{\Delta k}$$

`Graphics: Two waves initial initial Graphics: Two waves initial evolution NK/(1+K*K) NK/(1+K*K)**2`

Alternate view of two waves

Assuming that the shorter wave travels more slowly and the two waves start at time $t = 0$ being in phase at $x = 0$, the pattern will repeat exactly when the **previous** crests of each of the two waves match up precisely:



Two wave geometry. Solid=sum; dashed=longer, faster; dotted=shorter, slower

Thus we have

$$c_1 T = \frac{2\pi}{k_1} + c_g T \quad , \qquad c_2 T = \frac{2\pi}{k_2} + c_g T$$

We can think of these as simultaneous equations for $1/T$ and c_g:

$$c_g + \frac{2\pi}{k_1}\frac{1}{T} = c_1$$
$$c_g + \frac{2\pi}{k_2}\frac{1}{T} = c_2$$

solving these gives

$$c_g = \frac{c_1 \frac{2\pi}{k_2} - c_2 \frac{2\pi}{k_1}}{\frac{2\pi}{k_2} - \frac{2\pi}{k_1}} = \frac{c_1 k_1 - c_2 k_2}{k_1 - k_2} = \frac{\omega_1 - \omega_2}{k_1 - k_2} = \frac{\Delta\omega}{\Delta k}$$

Dispersion of group

If we consider the next order in our expansion for sharply peaked spectra

$$w \simeq \exp(\imath \mathbf{k}_0 \cdot \mathbf{x} - \imath \Omega(\mathbf{k}_0) t) \int d^3\mathbf{K}\, \phi(\mathbf{K}) \exp(\imath \mathbf{K} \cdot \epsilon \mathbf{x} - \imath \mathbf{K} \cdot \nabla_{\mathbf{k}} \Omega(\mathbf{k}_0) \epsilon t) \times$$

$$\exp(-\imath \frac{1}{2} \frac{\partial^2 \Omega}{\partial k_i \partial k_j} K_i K_j \epsilon^2 t)$$

In a frame moving with the group $\mathbf{X} = \epsilon \mathbf{x} - \mathbf{c}_g \epsilon t$, the changes on a time scale $\tau = \epsilon^2 t$ are determined by

$$w \simeq \exp(\imath \mathbf{k}_0 \cdot \mathbf{x} - \imath \Omega(\mathbf{k}_0) t) \int d^3\mathbf{K}\, \phi(\mathbf{K}) \exp(\imath \mathbf{K} \cdot \mathbf{X}) \exp(-\imath \frac{1}{2} \frac{\partial^2 \Omega}{\partial k_i \partial k_j} K_i K_j \tau)$$

For this we find the amplitude satisfies the Schrödinger equation

$$\frac{\partial}{\partial \tau} w = \frac{\imath}{2} \frac{\partial^2 \Omega}{\partial k_i \partial k_j} \frac{\partial}{\partial X_i} \frac{\partial}{\partial X_j} w$$

This looks like a diffusion equation (with an imaginary diffusivity) and can be solved in much the same way. In particular, we can look for Gaussian solutions

$$w = A(\tau) \exp(-\alpha_{ij}(\tau) X_i X_j)$$

The result can be seen in the 1D case

$$\frac{\partial}{\partial \tau} w = \frac{\imath}{2} \frac{\partial^2 \Omega}{\partial k^2} \frac{\partial^2}{\partial X^2} w$$

Plugging $w = A(\tau) \exp(-\alpha(\tau) X^2)$ into the previous equation and gathering the terms which are proportional to X^2 and to 1 gives

$$\frac{\partial}{\partial t} \alpha = -2\imath \Omega'' \alpha^2$$

$$\frac{\partial}{\partial t} A = -\imath \Omega'' \alpha A$$

This has solutions

$$\alpha = \frac{\alpha_0}{1 + 2\imath \alpha_0 \Omega'' t}$$

$$A = A(0) \sqrt{\frac{\alpha(t)}{\alpha_0}}$$