12.802 Small Scale Ocean Dynamics

Purpose:

Fundamental principles of GFD will be applied to study oceanic motions on scales from hundreds of kilometers down to centimeters. Topics will include wave motions, instabilities and turbulence.

Outline

1) Waves – using surface gravity waves as an example.
2) General linearized waves in rotating stratified fluid
3) Internal gravity waves
4) Boundaries, topography, and tides
5) Upwelling
6) River plumes
7) Instabilities
8) Turbulence

Basic wave dynamics and surface waves

Basic definitions:

The simplest form of a **wave** [a plane wave] has periodic variations in both space and time: for some property ψ we have

$$\psi = \psi_0 \cos(\mathbf{k} \cdot \mathbf{x} - \omega t)$$

Definitions:

- distances $\mathbf{x} = (x, y, z)$ representing (generally) east, north, and up
- wavenumbers $\mathbf{k} = (k, \ell, m)$
- wavelength $\lambda = 2\pi/|\mathbf{k}|$
- frequency ω [Note: you have probably seen this called "angular frequency" previously. We shall just call it the frequency, as is common in more advanced treatments.]
- period $T = 2\pi/\omega$ [Note the 2π factor arising from the use of angular frequency.]
- phase: we can define the phase as the argument of the cosine $\theta(\mathbf{x}, t) = \mathbf{k}\cdot\mathbf{x} - \omega t$. The signal has a maximum along the constant phase lines $\theta = n2\pi$, $n = \ldots, -2, -1, 0, 1, 2, \ldots$

Phase speed: Conceptually, if we put a ruler in the flow oriented in some direction, we can measure the rate at which the peaks and troughs move along this line. This implies $\theta(\mathbf{x}+c\hat{\mathbf{s}}\delta t, t+\delta t) = \theta(\mathbf{x}, t)$ where $\hat{\mathbf{s}}$ is a unit vector pointing along the ruler. For example, the rate of motion of a plane wave along the x axis satisfies

$$\mathbf{k}\cdot(\mathbf{x}+c^{(x)}\hat{\mathbf{x}}\delta t) - \omega(t+\delta t) = \mathbf{k}\cdot\mathbf{x} - \omega t$$

or

$$kc^{(x)} = \omega \quad , \quad c^{(x)} = \omega/k$$

Likewise $c^{(s)} = \omega/(\mathbf{k}\cdot\hat{\mathbf{s}})$. Note that $c^{(x)}, c^{(y)}, c^{(z)}$ are not the components of a vector; if we define

$$\hat{\mathbf{s}} = s_x\hat{\mathbf{x}} + s_y\hat{\mathbf{y}} + s_z\hat{\mathbf{z}}$$

then

$$c^{(s)} = \frac{\omega}{ks_x + \ell s_y + m s_z} \neq s_x\frac{\omega}{k} + s_y\frac{\omega}{\ell} + s_z\frac{\omega}{m}$$

Generalizations:

We shall also deal with cases where plane waves may be a reasonable description locally, but not on larger scales. In that case, we'll consider

$$\psi = A(\mathbf{x}, t)\exp(\imath\theta(\mathbf{x}, t)) \tag{1}$$

Since all physical fields must be real, we really mean either

$$\psi = \Re[A(\mathbf{x}, t)\exp(\imath\theta(\mathbf{x}, t))] \quad \textit{or} \quad \psi = A(\mathbf{x}, t)\exp(\imath\theta(\mathbf{x}, t)) + c.c.$$

For linear problems, we don't need to worry about which form we are using and we can work just with (1). But if we have to multiply two fields together, we will need to be careful either to take the real part before multiplying or include the conjugate solution. By analogy to the plane wave, we can think of the local wavenumber of the disturbance as $\nabla\theta$ and the local frequency as $-\frac{\partial}{\partial t}\theta$. We shall look at these forms in more detail later.
Graphics Variable k: `k and cos(theta)`

Dispersion relation:

The physics of wave problems enters into the **dispersion relation**: just as the frequency of a pendulum depends on various characteristics of the system, the frequency of a wave depends on properties of the medium and also on the wavenumber

$$\omega = \Omega(\mathbf{k}|g, H, N, f, \ldots)$$

The form of functional form of Ω will depend on the kind of wave we're considering; but many of the characteristics of wave propagation can be expressed in general in terms of Ω and its derivatives.

Linear equations:

The wave dynamics we shall deal with in this course are almost entirely linear; we assume that there is a steady background state (stratification, flow, etc.) on which are superimposed time-dependent, propagating wave disturbances. We assume that perturbations in velocities associated with the waves are weak so that

$$\epsilon \equiv \frac{|\mathbf{u}'|}{c} << 1$$

In the simple case where the background state has no flow, we can compare the terms in the advective operator

$$\begin{aligned} \frac{D}{Dt} &= \frac{\partial}{\partial t} \ + \ \mathbf{u}' \cdot \nabla \\ &\sim \omega \qquad\quad |\mathbf{u}'||\mathbf{k}| \\ &\sim c|\mathbf{k}| \quad\;\; |\mathbf{u}'||\mathbf{k}| \\ &\sim c|\mathbf{k}| \qquad \epsilon c|\mathbf{k}| \end{aligned}$$

Thus for small amplitude waves, we can replace

$$\frac{D}{Dt} \longrightarrow \frac{\partial}{\partial t}$$

We will also linearize other terms as needed.