1. Seiches and resonances

a) Consider a circular, flat-bottomed lake of depth H and radius $R >> H$. What are the natural oscillation frequencies? Use this to estimate the period for the gravest mode in Lake Victoria and Great Salt Lake.

b) Consider the tidally forced problem in a deep ocean/shelf geometry but ignoring f. The shelf has depth H_1 and a vertical wall at $x = -L$. The "shelf break" is at $x = 0$; the deep ocean $H = H_2$ extends infinitely far to positive x. The tidal forcing is $\eta_e = \eta_0 \cos(kx + \omega t)$. Find $|\eta(-L)|/\eta_0$. Discuss the resonance and its relation to an approximate free mode. Consider what happens as $H_2 >> H_1$.

2. Tidally-forced mean flows

Consider a 2D bank with shallow water dynamics including bottom drag

$$
\begin{aligned}
\frac{\partial}{\partial t}u + u\frac{\partial}{\partial x}u - fv &= -g\frac{\partial}{\partial x}\eta - ru/h \\
\frac{\partial}{\partial t}v + u\frac{\partial}{\partial x}v + fu &= -rv/h \\
\frac{\partial}{\partial t}\eta + \frac{\partial}{\partial x}uh &= 0 \\
h &= H(x) + \eta(x,t)
\end{aligned}
$$

The approach here is somewhat different from the version in class.

a) Write the equations for the transport $U = uh$, $V = vh$. Consider the tidal forcing as applying a boundary condition $U \to A\cos\omega t$ as $x \to \pm\infty$ assuming H becomes constant.

b) Derive the time averages of these equations; what does the mass equation tell you about $\overline{U}$? The y-momentum equation gives you an expression for $\overline{v}$. Express this in terms of $\overline{huv}$.

c) Now, we want to calculate the Reynolds' momentum flux. Suppose the wave amplitude is small and the mean flows are second order. Write the lowest order equations. Apply the tidally forcing $u_0 H = A\cos\omega t$. Find v_0 and estimate $\overline{huv} \simeq H\overline{u_0 v_0}$.

d) Use this to write an expression for $\overline{v}$. For a ridge topography, sketch the resultant velocity and show that is anticyclonic.

3. Wind-driven upwelling

Consider a stratified ocean with a wind stress

$$\tau/\rho = T\cos(\ell y)$$

- Find the Ekman pumping. When would the stratification near the surface matter? Figure out the conditions but for the rest of the problem assume it does not.
- Find the interior response and sketch the flow pattern.

4. Temperature and salinity eddy fluxes.

The tracer conservation equation for a generic tracer c is

$$\frac{\partial c}{\partial t} + \nabla \cdot \mathbf{u}c = \kappa_c \nabla^2 c$$

where κ_c is the molecular diffusivity of that tracer.

1) Assuming that c can be split into a large scale component $\overline{c}$ and a small scale component c', so that $c = \overline{c} + c'$ and $\overline{c'} = 0$, and similarly for the velocity field, write down equations for the time-evolution of the large scale temperature $\overline{T}$ and salinity $\overline{S}$.
2) Assume that the small-scale dynamics influence the large scale fields only through vertical fluxes, which can be parameterized in terms of diffusion down the large scale gradient with eddy diffusivity κ_T^* for temperature and κ_S^* for salinity. Rewrite the equations from (a) incorporating this parameterization of the small-scale fluxes.
3) Now combine both equations from (b) to form one equation for the large scale buoyancy $\overline{b} = \alpha\overline{T} - \beta\overline{S}$, and show that the small scale fluxes of buoyancy can again be written in terms of diffusion in the direction of the large scale density gradient, with eddy diffusivity
$$\kappa_b^* = \frac{\kappa_T^* R_\rho - \kappa_S^*}{R_\rho - 1}$$
where $R_\rho = \frac{\alpha \partial \overline{T}/\partial z}{\beta \partial \overline{S}/\partial z}$.
4) For warm salty water overlying cold fresh water, small-scale fluxes of salt and temperature may be due to either turbulent mixing, or salt-fingering. In the turbulent regime, salt and heat are mixed equally efficiently. In the salt fingering regime $(\alpha\overline{w'T'})/(\beta\overline{w'S'}) \approx 0.7$. Find κ_b^*/κ_S^* under these two circumstances. Comment on the sign of κ_b^*

5. Instability of an oscillatory shear flow.

In class we discussed the Kelvin-Hemholtz instability of a shear flow in a stably stratified fluid and used that model to study the instability of inertia-gravity waves. Here we wish to step back and investigate whether the oscillatory character of the shear associated with inertia-gravity waves may play an important role in the instability.

Consider the same setup we discussed in class. The basic state consists of two fluids of different densities $\rho_1 > \rho_2$ separated by an interface. The lighter fluid is on top of the heavier one, so the stratification is stable. The flow in the upper layer is U_2 and in the lower layer U_1. While U_1 and U_2 are constant in space in each layer, they are different in the two layers giving rise to a shear at the interface. Departing from the analysis we discussed in class, we will assume that the flow in the two layers oscillates in time so that $U_1 = U_1(t)$ and $U_2 = U_2(t)$.

1) Write the evolution equation for small perturbations of the interface $\eta(x,t)$, i.e. perturbations independent of the y-direction. Consider solutions of the form $\eta(x,t) = \hat{\eta}(t)\exp(kx)$. Show that the evolution equation for $\hat{\eta}(t)$ is

$$\frac{d^2\hat{\eta}}{d^2t} + 2ik(\alpha_1 U_1 + \alpha_2 U_2)\frac{d\hat{\eta}}{dt} +$$

$$+\left(gk(\alpha_1-\alpha_2)-\alpha_1k^2U_1-\alpha_2k^2U_2+i\alpha_1k\frac{dU_1}{dt}+i\alpha_2k\frac{dU_2}{dt}\right)\hat{\eta}=0$$

where

$$\alpha_1=\frac{\rho_1}{\rho_1+\rho_2},\qquad \alpha_2=\frac{\rho_2}{\rho_1+\rho_2}.$$

2) Show that you can eliminate the first derivative term by means of the substitution,

$$\zeta(t)=\hat{\eta}\exp\left(-ik\int(\alpha_1U_1+\alpha_2U_2)\mathrm{d}t\right)$$

Verify that the stability of $\zeta(t)$ is governed by,

$$\frac{d^2\zeta}{d^2t}+\left(gk(\alpha_1-\alpha_2)-\alpha_1\alpha_2k^2(U_1-U_2)^2\right)\zeta=0$$

3) Now assume that

$$U_1-U_2\equiv\left(1-\frac{\alpha_2}{\alpha_1}\right)\Delta U\cos\omega t$$

as a simple model of an oscillatory wave field. Show that in this limit the stability of the interface is governed by an equation of Mathieu type,

$$\frac{d^2\zeta}{d^2t}+(\delta+\epsilon+\epsilon\cos 2\omega t)\,\zeta=0.$$

Express the parameters δ and ϵ in terms of α_1, α_2 and ΔU.

4) Do you think the addition of oscillations makes the interface more or less stable compared to the classical Kelvin-Helmoltz problem with $\omega=0$? You may want to read about Mathieu's equation to answer this question.