12.802 Small-scale ocean dynamics

Problem set 2. Due date: April 17, 2018

1. **Parcel arguments and the minimum frequency of inertia-gravity waves.** In class we showed that a water parcel in a stratified fluid oscillates at a frequency $\omega = N\cos\theta$ if displaced from equilibrium, where N^2 is the stratification frequency and θ the angle to the horizontal along which the parcel oscillates. See if you can extend the argument to the case when the stratified fluid is rotating. At what frequency will the parcel oscillate?

2. **Radiation of topographic waves in the presence of rotation and damping.** In class we discussed the radiation of topographic waves in a stratified fluid by a constant barotropic mean flow U_0 and a sinusoidal topography of the form $h(x) = h_0 \cos kx$. You are now asked to repeat the same calculation but including rotation and linear damping. Show that under the same approximations made in class, the problem is described by this set of equations,

$$u_t + U_0 u_x - fv = -p_x - ru, \qquad v_t + U_0 v_x + fu = -p_y - rv, \tag{1}$$
$$w_t + U_0 w_x = -p_z + b - rw, \qquad b_t + U_0 b_x + N^2 w = -rb, \tag{2}$$
$$u_x + v_y + w_z = 0, \tag{3}$$

with the bottom boundary condition $w = U_0 h_x$ at $z = 0$. The notation is the same as that used in class. U_0 and N are constant. r is a linear and constant drag coefficient.

   (a) Derive an equation for the vertical velocity. Find solutions corresponding to radiating waves. Discuss how you choose the appropriate boundary condition at $z \to \infty$. How does the addition of a small r modify the solutions compared to the case studied in class? A sketch of the solution may be useful to answer this question.

   (b) Compute the vertical energy flux. Show that in the limit $f = r = 0$, the vertical energy flux reduces to the solution derived in class. Discuss why the limit $r \to 0$ is useful to understand why wave solutions result in an upward energy flux.

3. **Reflection of inertia-gravity waves.** Despite the examples in text-books, the ocean does not have vertical sidewalls. The continental rise, slope and shelf topographies and the sides of seamounts all lie at a shallow angle to horizontal, with slopes, α, typically 10^{-3} to 10^{-2}; occasionally 10^{-1}.
Internal waves with uniform density stratification N have their frequency determined by the angle of propagation (with respect to the horizontal direction), and not by the wavelength. More usually waves tend to reflect from a solid, plane boundary such that the incident and reflected waves have wavevectors $\boldsymbol{k}$ symmetric about the normal to the boundary. However if that were true for internal waves, the frequency ω would differ for incident and reflected waves

at a sloping boundary: yet the separable solutions of the linear wave equation show that the frequency should remain the same for the two waves. The only possibility is that they reflect with $\boldsymbol{k}$ symmetric about the vertical, regardless of the angle α!

The boundary condition, neglecting viscous boundary layers, is that the normal component of velocity vanishes at the boundary.

Given an incident wave with vertical velocity $w = A_i \exp(ik_i x + im_i z - i\omega_i t)$:

(a) Solve for the amplitude A_r and wavenumbers (k_r, m_r) of the reflected wave $w_r = A_r \exp(ik_r x + im_r z - i\omega_r t)$ in terms of A_i, k_i and m_i. The wave equation is

$$\partial_t^2(\partial_x^2 w + \partial_z^2 w) + N^2 \partial_x^2 w = 0$$

neglecting rotation. The boundary condition is

$$\boldsymbol{u} \cdot \hat{\boldsymbol{n}} = 0 \qquad \rightarrow \qquad w = u \tan\alpha$$

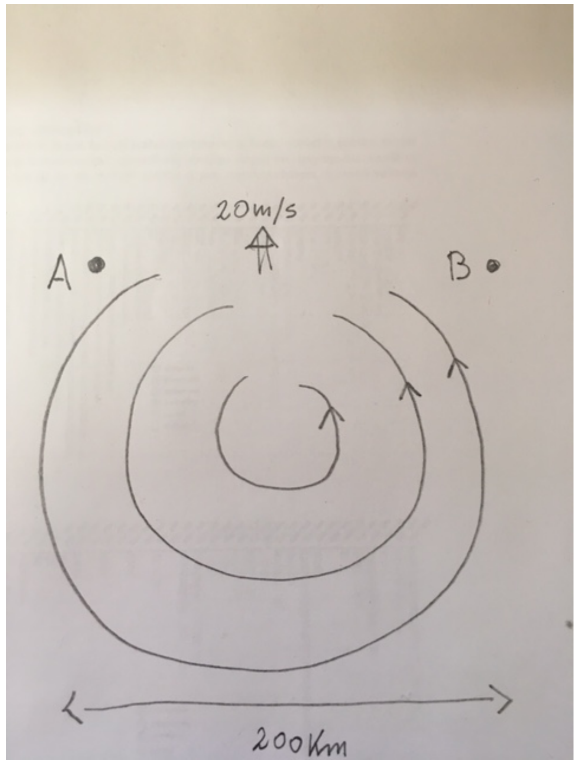
at the solid boundary, $z = x \tan\alpha$. In order that this be satisfied everywhere along the boundary, the wavenumber component parallel with the boundary, as well as the frequency must be the same for the two waves.

(b) Show how the reflected wave amplitude and energy density becomes very large for some values of bottom slope α. How does the reflected wavelength compare with the incident wavelength?

(c) Show that the vorticity of the reflected wave near the wall, which is just the velocity gradient normal to the wall, also becomes large, tending to cause shear-flow instability and mixing.

(d) Because the tides have a lot of energy, it is interesting to ask what values of bottom slope α will cause the lunar semi-diurnal and solar diurnal frequencies to produce large amplitude reflected waves (and hence mixing). Choose $N = 2\pi/(1/2 \text{ hour})$. Bonus: It would be interesting to make a map comparing the observed bottom slope with the critical slope for these frequencies, using observed values of $N(z)$.

4. **Waves in hurricane wake.** Consider a hurricane passing over the two moorings marked as A and B in the sketch below. The hurricane generates a cyclonic wind stress on the ocean as marked by the arrows in the sketch. The hurricane has a radius of 100 km and drifts northward at approximately 20 m/s. Sketch the temporal evolution of the wind stress above the two moorings, paying particular attention at how the stress veers over time. Discuss whether you expect to observe more inertial-gravity wave energy at mooring A or B in response to the passing hurricane. Assume that $f = 10^{-4}$ s^{-1}.



5. **Inertial waves and Taylor column stiffness.** In class we have seen that the inertia gravity wave equations also describe important steady-flow problems (e.g., flow over mountains). For the most part we studied internal waves with density stratification (N) and rotation (f) in the limit $N \gg f$. Consider now the case $N = 0$, (constant density ρ), which still allows inertial waves due to the Coriolis effect, yet they have somewhat peculiar properties. Suppose there is no variation of any quantity in the y-direction (though there is a velocity v in that direction). The wave equation for the vertical velocity becomes

$$(w_{xx} + w_{zz})_{tt} + f^2 w_{zz} = 0.$$

   (a) Find the dispersion relation $\omega(k, m)$ for waves of the form

$$w = \text{Real}\left(De^{ikx+imz-i\omega t}\right)$$

   and make a sketch of it.

   (b) Describe the way the angle of the wave crests and the direction of the group velocity vary with ω. Suppose the waves are generated by a moving boundary at $z = 0$ and radiate upward into the fluid which lies above $z = 0$. The boundary condition is

$$w = \text{Real}\left(Ae^{ik_0x-i\omega_0 t}\right) = A\cos(k_0x - \omega_0 t), \qquad \text{at } z = 0.$$

   The linearized equations for this problem are

$$u_t - fv = -p_x/\rho; \qquad v_t + fu = -p_y/\rho = 0;$$
$$w_t = -p_z/\rho; \qquad u_x + w_z = 0.$$

The pressure is $p = p_0(z)+p(x,z,t)$ where p_0 is the background hydrostatic pressure. $\rho =$ constant.

(c) Find the u, v, and w velocity components for this wave solution. Note the pressure field p is not hydrostatic. Show that if the frequency is much less than f, $\omega_0 \ll f$, the velocity field begins to resemble that of geostrophic Taylor columns, that is, a 'flow' even though these are propagating waves.

(d) Discuss some (not all) of the following: the magnitude of the vertical group velocity that describes the propagation upward into the fluid; the amplitudes of u, v and w; the vertical-vorticity balance for this flow; the energy flux upward into the fluid; how the amplitude of the pressure force exerted by the boundary depends on the Coriolis frequency, f.