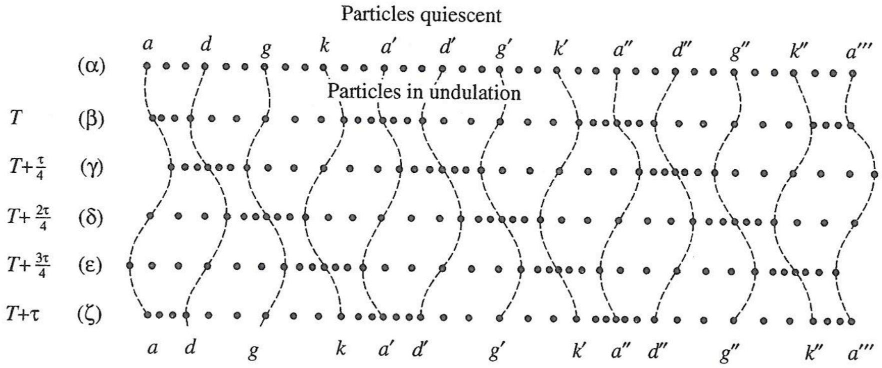


1. In the topmost line (α), the elements of air, represented by points a, b, c, d, etc., are placed at equal distances in air that is still (quiescent).
2. In the next line (β), the elements are assumed (arbitrarily) to be distributed, as shown, condensed as at a, a' etc., and expanded as at g, g', etc.
3. In the third line (γ), the points of condensation occur at about d, d', etc., while points of rarefaction occur at k, k', etc.
4. In the fourth line (δ), the states of condensation and of rarefaction have travelled a little further along in the same direction; and so on in the successive lines (ε) and (ζ).

The basic idea in this conception of a wave is that of a *state of displacement* travelling continuously in one direction, while the motions of the individual elements are small and oscillatory. And the major step that Newton took beyond formulating this conception is to demonstrate (as he does in Propositions XLVII and XLIX) how the condensations and rarefactions produced by such oscillatory motions can be maintained consistently with the elasticity properties of air. Simple as these notions are, it is perhaps not possible for us to realize the novelty and the boldness of the conception and the difficulty in formulating analytically the underlying ideas.

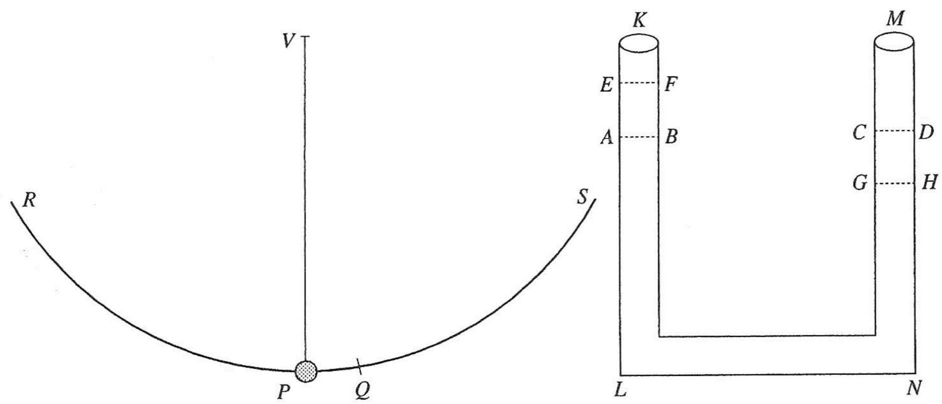
152. Propositions XLIV–XLVI: the propagation of long waves in canals

In Proposition XLIV, Newton considers an extremely simple example of oscillation of liquid columns in a U-shaped canal; and by a stroke of intuition extends it to solve the problem of long waves in canals. These propositions are explained in so exemplary a fashion that one cannot do better than quote him verbatim. (The crucial arguments are underlined.)

Proposition XLIV. Theorem XXXV

If water ascend and descend alternately in the erected legs KL, MN of a canal or pipe; and a pendulum be constructed whose length between the point of suspension and the centre of oscillation is equal to half the length of the water in *the canal: I say, that the water will ascend and descend in the same times in which the pendulum oscillates.*

I measure the length of the water along the axes of the canal and its legs, and make it equal to the sum of those axes; and take no notice of the resistance of the water arising from its attrition by the sides of the canal. Let, therefore, *AB*, *CD* represent the mean height of the water in both legs; and when the water in the leg *KL* ascends to the height *EF*, the water will descend in the leg *MN* to the height *GH*. Let *P* be a pendulous body, *VP* the thread, *V* the point of



suspension, *RPQS* the cycloid which the pendulum describes, *P* its lowest point, *PQ* an arc equal to the height *AE*. The force with which the motion of the water is accelerated and retarded alternately is the excess of the weight of the water in one leg above the weight in the other; and, therefore, when the water in the leg *KL* ascends to *EF*, and in the other leg descends to *GH*, that force is double the weight of the water *EABF*, and therefore is to the weight of the whole water as *AE* or *PQ* to *VP* or *PR*. The force also with which the body *P* is accelerated or retarded in any place, as *Q*, of a cycloid, is (by Cor., Prop. LI, Book I) to its whole weight as its distance *PQ* from the lowest place *P* to the length *PR* of the cycloid. Therefore the motive forces of the water and pendulum, describing the equal spaces *AE*, *PQ*, are as the weights to be moved; and therefore if the water and pendulum are quiescent at first, those forces will move them in equal times, and will cause them to go and return together with a reciprocal motion. Q.E.D.

COR. I. Therefore the reciprocations of the water in ascending and descending are all performed in equal times, whether the motion be more or less intense or remiss.

COR. II. If the length of the whole water in the canal be of $6\frac{1}{9}$ feet of French measure, the water will descend in one second of time, and will ascend in another second, and so on by turns *in infinitum*; for a pendulum of $3\frac{1}{18}$ such feet in length will oscillate in one second of time.

COR. III. But if the length of the water be increased or diminished, the time of the reciprocation will be increased or diminished as the square root of the length.

Proposition XLV. Theorem XXXVI

The velocity of waves varies as the square root of the breadths.

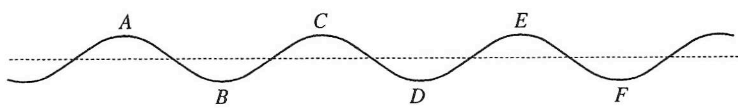
This follows from the construction of the following proposition.

Proposition XLVI. Problem X

To find the velocity of waves.

Let a pendulum be constructed, whose length between the point of suspension and the centre of oscillation is equal to the breadth of the waves, and in the time that the pendulum will perform one single oscillation the waves will advance forwards nearly a space equal to their breadth.

That which I call the breadth of waves is the transverse measure lying between the deepest part of the hollows, or the tops of the ridges. Let *ABCDEF* represent the surface of stagnant water ascending and descending in successive



waves; and let *A*, *C*, *E*, etc., be the tops of the waves; and let *B*, *D*, *F*, etc., be the intermediate hollows. Because the motion of the waves is carried on by the successive ascent and descent of the water, so that the parts thereof, as *A*, *C*, *E*, etc., which are highest at one time, become lowest immediately after; and because the motive force, by which the highest parts descend and the lowest ascend, is the weight of the elevated water, that alternate ascent and descent will be analogous to the reciprocal motion of the water in the canal, and will observe the same laws as to the times of ascent and descent; and therefore (by Prop. XLIV) if the distances between the highest places of the waves *A*, *C*, *E* and the lowest *B*, *D*, *F* be equal to twice the length of any pendulum, the highest parts *A*, *C*, *E* will become the lowest in the time of one oscillation, and in the time of another oscillation will ascend again. Therefore with the passage of each wave, the time of two oscillations will occur; that is, the wave will describe its breadth in the time that pendulum will oscillate twice; but a pendulum of four times that length, and therefore equal to the breadth of the waves, will just oscillate once in that time. Q.E.I.

COR. I. Therefore, waves, whose breadth is equal to $3\frac{1}{18}$ French feet, will advance through a space equal to their breadth in one second of time; and therefore in one minute will go over a space of $183\frac{1}{3}$ feet; and in an hour a space of 11000 feet, nearly.

COR. II. And the velocity of greater or less waves will be augmented or diminished as the square root of their breadth.

These things are true upon the supposition that the parts of water ascend or descend in a straight line; but, in truth, that ascent and descent is rather performed in a circle; and therefore I give the time defined by this proposition as only approximate.

The theory of long waves in canals

It is clear that in Proposition XLVI, Newton, in effect, considers the propagation of long waves in canals, that is, of waves of infinitesimal height (or, amplitude) travelling along a straight canal with a horizontal bed and vertical sides.

Let the x-axis be taken along the length of the canal and the y-axis in the vertical upward direction. We shall suppose that the motion takes place in the two dimensions x and y. Let the ordinate of the free surface at x and at time t be

$$y_0 + \eta \quad \text{where } y_0 \text{ specifies the undisturbed surface.} \tag{1}$$

We shall suppose that the pressure at any point (x, y) is sensibly the same as the static pressure at that depth, namely,

$$p - p_0 = g\rho(y_0 + \eta - y), \tag{2}$$

where p_0 denotes the uniform pressure above the free surface $(y \geqslant y_0)$ and g denotes the value of gravity. Under these assumptions, the equation of hydrostatic equilibrium is

$$\frac{\partial p}{\partial x} = g\rho\,\frac{\partial \eta}{\partial x}. \tag{3}$$

It follows that the horizontal velocity u is a function of x and y only since the horizontal component of the acceleration is the same in a plane perpendicular to the x-direction.

The equation governing the horizontal motion is

$$\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} = -\frac{1}{\rho}\,\frac{\partial p}{\partial x}. \tag{4}$$

If the velocities, as we shall suppose, are sufficiently small, we can neglect the non-linear inertial term, and we can write

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\,\frac{\partial p}{\partial x} = -g\,\frac{\partial \eta}{\partial x}. \tag{5}$$

It is convenient to introduce the Lagrangian displacement, $\xi(x, t)$, defined by

$$u(x, t) = \frac{\partial}{\partial t}\xi(x, t). \tag{6}$$

The equation of continuity can be obtained by ascertaining the volume of the fluid which has, up to the time t, entered the space bounded by the planes x and $x + \delta x$; thus,

$$-\delta x \frac{\partial}{\partial x}(\xi h b) = \eta b \delta x \tag{7}$$

where h and b denote the height and the breadth of the canal. Therefore,

$$\eta = -h\frac{\partial \xi}{\partial x}, \tag{8}$$

or,

$$\frac{\partial \eta}{\partial t} = -h\frac{\partial u}{\partial x} \quad \text{by equation (6).} \tag{9}$$

Equations (5) and (9) together give,

$$\frac{\partial^2 \eta}{\partial t^2} = gh\frac{\partial^2 \eta}{\partial x^2}. \tag{10}$$

Defining the velocity,

$$c = \sqrt{(gh)}, \tag{11}$$

equation (10) takes the standard form,

$$\frac{\partial^2 \eta}{\partial t^2} = c^2\frac{\partial^2 \eta}{\partial x^2}. \tag{12}$$

As is well known, the general solution of this equation is

$$\eta = f(x - ct) + k(x + ct), \tag{13}$$

where f and k are arbitrary functions of the arguments specified. In particular, a solution, representing a progressive wave of frequency ω and wavelength λ is given by

$$\eta = \eta_0 \cos\left(\omega t - \frac{2\pi}{\lambda}x\right), \tag{14}$$

where

$$\omega^2 = \frac{4\pi^2}{\lambda^2}c^2 = \frac{4\pi^2}{\lambda^2}gh \quad \text{by equation (11).} \tag{15}$$

Therefore,

$$\text{period of oscillation } (=2\pi/\omega) \times \text{velocity } (=\sqrt{(hg)}) = \text{wavelength } (=\lambda). \tag{16}$$

This is the relation that Newton states at the conclusion of Proposition XLVI (without any calculation!). His concluding remark: 'the time defined by the proposition is only approximate', refers to the strictly infinitesimal amplitudes to which Proposition XLVI applies in contrast to Proposition XLV which is *exact* by the comparison with a cycloidal pendulum.