Upwelling

We look at the two-dimensional case with $\frac{\partial}{\partial y} = 0$ and assume that the only important mixing or friction terms act vertically.

$$\begin{aligned}
\frac{D}{Dt}u - fv &= -\frac{\partial}{\partial x}p + \frac{\partial}{\partial z}\nu\frac{\partial}{\partial z}u \\
\frac{D}{Dt}v + fu &= \frac{\partial}{\partial z}\nu\frac{\partial}{\partial z}v \\
\frac{D}{Dt}w &= -\frac{\partial}{\partial z}p + b + \frac{\partial}{\partial z}\nu\frac{\partial}{\partial z}w \\
\frac{D}{Dt}b &= \frac{\partial}{\partial z}\kappa\frac{\partial}{\partial z}b \\
\frac{\partial}{\partial x}u + \frac{\partial}{\partial z}w &= 0
\end{aligned}$$

We look at $\frac{D}{Dt}$ of the horizontal vorticity $q = u_z - w_x$ and the streamfunction $u = \phi_z$, $w = -\phi_x$, $q = \nabla^2\phi$. We then get three equations (along with the Poisson equation for ϕ)

$$\begin{aligned}
\frac{D}{Dt}q &= fv_z - b_x + \frac{\partial}{\partial z}\nu\frac{\partial}{\partial z}q + \frac{\partial}{\partial z}(\nu_z\phi_{zz}) \\
\frac{D}{Dt}v &= -f\phi_z + \frac{\partial}{\partial z}\nu\frac{\partial}{\partial z}v \\
\frac{D}{Dt}b &= \frac{\partial}{\partial z}\kappa\frac{\partial}{\partial z}b \\
\frac{D}{Dt}\psi &= \frac{\partial}{\partial t}\psi + \phi_z\frac{\partial}{\partial x}\psi - \phi_x\frac{\partial}{\partial z}\psi \equiv \frac{\partial}{\partial t}\psi + [\phi, \psi]
\end{aligned}$$

Note that the forcing for q is basically the imbalance in the thermal wind.

Interior/ boundary layer split

The friction is important in a thin layer near the top or bottom. Let's take the original momentum equations and separate them into interior and boundary layer flow $\mathbf{u} = \mathbf{u}_i(\mathbf{x}, z, t) + \mathbf{u}_e(\mathbf{x}, z', t)$ with z varying on a scale H and z' on an Ekman layer scale $d << H$. We scale assumeing $\mathbf{u}_i$ and $\mathbf{u}_e$ are order U, $\mathbf{x} \sim L$, $t \sim L/U$, $p \sim fUL$; then

$$\frac{D}{Dt}\mathbf{u}_i + \frac{D}{Dt}\mathbf{u}_e + f\hat{\mathbf{z}} \times \mathbf{u}_i + f\hat{\mathbf{z}} \times \mathbf{u}_e = -\nabla p + \frac{\partial}{\partial z}\nu\frac{\partial}{\partial z}\mathbf{u}_i + \frac{\partial}{\partial z'}\nu\frac{\partial}{\partial z'}\mathbf{u}_e$$

and we can take the $\mathbf{u}_i$ to be inviscid

$$\boxed{\frac{D}{Dt}\mathbf{u}_i + f\hat{\mathbf{z}} \times \mathbf{u}_i = -\nabla p}$$

leaving

$$
\begin{array}{cccc}
\frac{D}{Dt}\mathbf{u}_e + f\hat{\mathbf{z}}\times\mathbf{u}_e = & \frac{\partial}{\partial z'}\nu\frac{\partial}{\partial z'}\mathbf{u}_e + & \frac{\partial}{\partial z}\nu\frac{\partial}{\partial z}\mathbf{u}_i \\
\frac{U^2}{L} \quad fU & \frac{\nu U}{d^2} & \frac{\nu U}{H^2} \\
\frac{U}{fL} \quad 1 & \frac{\nu}{fd^2} & \frac{\nu}{fH^2}
\end{array}
$$

where the second equation sohw the ratio to the Coriolis term. For small Rossby number and small Ekman number ν/fH^2 we have to take $d \sim \sqrt{\nu/f}$ to acheive a balance

$$
\boxed{f\hat{\mathbf{z}}\times\mathbf{u}_e = \frac{\partial}{\partial z}\nu\frac{\partial}{\partial z}\mathbf{u}_e}
$$

The boundary conditions will be met by the combination of the two flow fields.

Interior

Away from the boundaries, the frictional terms can be neglected, and the flow is nearly in thermal wind balance. Thus we can take

$$
fv_z = b_x \quad \Rightarrow \quad fv = p_x \ , \quad b = p_z
$$

(dropping the interior subscript and going back to the $\frac{\partial}{\partial y} = 0$ form). Th flow is determined by the "omega equation". This is derived by taking $f\frac{\partial}{\partial z}$ of the meridional momenum equations and subtracting $\frac{\partial}{\partial x}$ of the buoyancy equation, i.e.

$$
\begin{aligned}
\frac{\partial}{\partial z}[\frac{\partial}{\partial t}p_x + [\phi, p_x] + f^2\phi_z] - \frac{\partial}{\partial x}[\frac{\partial}{\partial t}p_z + [\phi, p_z] - N^2\phi_x] &= 0 \\
\text{or} \quad [\phi_z, p_x] - [\phi_x, p_z] + f^2\phi_{zz} + N^2\phi_{xx} &= 0 \\
&\Rightarrow \\
(f^2 + p'_{xx})\phi_{zz} + \phi_{xx}(N^2 + p'_{zz}) - 2\phi_{xz}p'_{xz} &= 0
\end{aligned}
$$

where we've specifically built in the stratification using $p = \int^z B(z) + p'$ with $N^2 = B_z$. For $N^2H^2 \sim f^2L^2$ and small Rossby number $v << fL$, the p' terms are small, and we just have

$$
\boxed{f^2\phi_{zz} + N^2\phi_{xx} = 0}
$$

The boundary conditions will be set by the frictional layers and the bottom slope.

Ekman layers

We can write the Ekman layer equations in terms of a complex velocity

$$U_e \equiv u_e + \imath v_e$$

as

$$\frac{\partial}{\partial z}\nu\frac{\partial}{\partial z}U_e = \imath f U_e$$

Surface

At the top, the boundary conditions are just

$$\nu\frac{\partial}{\partial z}U_e = \tau^x + \imath\tau^y$$

since $\frac{\partial}{\partial z}\mathbf{u}_i$ is order d/H smaller than the boundary layer terms. Here $\vec{\tau}$ is the wind stress divided by the water density. Integrating the U_e equation and taking the boundary layer velocities to vanish as $z << 0$ gives

$$\imath f\int dz U_e = \tau^x + \imath\tau^y \quad \text{or} \quad \int \mathbf{u}_e = -\hat{\mathbf{z}} \times \frac{\vec{\tau}}{f}$$

Taking the divergence of this equation leads to the Ekman pumping/ suction equation

$$\int \nabla\cdot\mathbf{u}_e = -\int\frac{\partial}{\partial z}w_e = -w_e(0) = w_i(0) = \frac{\partial}{\partial x}\frac{\tau^y}{f} - \frac{\partial}{\partial y}\frac{\tau^x}{f} = \mathrm{curl}\frac{\vec{\tau}}{f}$$

For a wind in the y direction, then,

$$-\phi_x(0) = \frac{\partial}{\partial x}\frac{\tau^y}{f} \quad \Rightarrow \quad \phi = -\frac{\tau^y}{f}$$

To regularize the problem, we assume that τ increases from zero right at the coast to T offshore. Then ϕ goes from zero to $-T/f$; $\phi_x < 0$ implying upwelling into the surface layer to accomodate the increasing offshore Ekman transport.

Example: Suppose we have a very deep fjord with a wind blowing along it. Idealize this as $\tau^y = \tau_0\sin(\pi x/L)$. Then the interior solution is

$$\phi = -\frac{\tau_0}{f}\sin(\pi x/L)\exp\left(\frac{N\pi}{fL}z\right)$$

and the compensating eastward transport occuring in the surface boundary layer.

Bottom

The top condition is not affected by the form $\nu(z)$, but the bottom layer, which must satisfy $\mathbf{u}_i + \mathbf{u}_e = 0$ at $z = -H(x)$ is. We will take ν to be constant so that we can write a solution

$$U_e = U_{e0} \exp(-\sqrt{\frac{\imath f}{\nu}}[z+H]) = U_{e0} \exp(-(1+\imath)(z+H)/d)$$

with $d = \sqrt{2\nu/f}$ the Ekman layer thickness. The boundary condition implies

$$U_{e0} = -u_i(\mathbf{x}, -H) - \imath v_i(\mathbf{x}, -H) = -U_i(\mathbf{x}, -H)$$

The mass equation for the Ekman vertical velocity is

$$\frac{\partial}{\partial z} w_e = -\nabla \cdot \mathbf{u}_e = -\Re\left([\frac{\partial}{\partial x} - \imath \frac{\partial}{\partial y}]\, U_e \right) = \Re\left([\frac{\partial}{\partial x} - \imath \frac{\partial}{\partial y}]\, U_i(\mathbf{x}, -H) E \right)$$

with $E = \exp(-(1+\imath)(z+H)/d)$. We can integrate this to find

$$-w_e(\mathbf{x}, -H) = w_i(\mathbf{x}, -H) = \Re\left(\int_{-H}^{\infty} [\frac{\partial}{\partial x} - \imath \frac{\partial}{\partial y}]\, U_i(\mathbf{x}, -H) E \right)$$

or

$$\begin{aligned}
w_i(\mathbf{x}, -H) &= \Re\left([\frac{\partial}{\partial x} - \imath \frac{\partial}{\partial y}] \int_{-H}^{\infty} U_i(\mathbf{x}, -H) E \right) + \Re\left(-U_i(\mathbf{x}, -H)\, [\frac{\partial}{\partial x} - \imath \frac{\partial}{\partial y}]\, H \right) \\
&= \Re\left(\frac{d}{1+\imath} [\frac{\partial}{\partial x} - \imath \frac{\partial}{\partial y}]\, U_i(\mathbf{x}, -H) \right) - \mathbf{u}_i(\mathbf{x}, -H) \cdot \nabla H \\
&= \Re\left(\frac{d}{2}(1-\imath) [\frac{\partial}{\partial x} - \imath \frac{\partial}{\partial y}]\, U_i(\mathbf{x}, -H) \right) - \mathbf{u}_i(\mathbf{x}, -H) \cdot \nabla H
\end{aligned}$$

Working out the terms shows us that

$$[\frac{\partial}{\partial x} - \imath \frac{\partial}{\partial y}]\, U_i(\mathbf{x}, -H) = \nabla \cdot \mathbf{u}_i + \imath \zeta_i - \mathbf{u}_z \cdot \nabla H - \imath (v_z H_x - u_z H_y)$$

The last two terms will be negligible because the interior shear is weak; therefore, we end up with

$$w_i(\mathbf{x}, -H) = \frac{d}{2} \nabla \cdot \mathbf{u}_i + \frac{d}{2} \zeta_i - \mathbf{u} \cdot \nabla H$$

The interior divergence is small, so that w_i has a contribution from the topographic slope and the Ekman pumping associted with the bottom stress acting against the interior vortical motion

$$w_i(\mathbf{x}, -H) = \frac{d}{2} \zeta_i - \mathbf{u} \cdot \nabla H$$

For our $\frac{\partial}{\partial y} = 0$ problem, this just gives

$$\phi_x(x, -H) = -\frac{d}{2} v_x + \phi_z(x, -Ha) H_x$$

$$w_e(\mathbf{x}, -H) = -w_i(\mathbf{x}, -H) = \Re\left([\frac{\partial}{\partial x} - \imath\frac{\partial}{\partial y}]\, U_i(\mathbf{x}, -H)E\right)$$

The large term here will come from the $1/d$ in the exponent so that

$$\frac{\partial}{\partial z}w_e \simeq \Re\left(U_{e0}\frac{1+\imath}{d}(H_x - \imath H_y)\exp(-(1+\imath)(z+H)/d)\right)$$

The total horizontal velocity is

$$U = U_i(\mathbf{x}, z) - U_i(\mathbf{x}, -H)\exp(-(1+\imath)(z+H)/d)$$

where we have defined the complex interior velocity and satisfied the bottom condition on the horizontal velocities. We now use

$$w_z = -\nabla\cdot\mathbf{u} = -\Re\left([\frac{\partial}{\partial x} - \imath\frac{\partial}{\partial y}]U\right)$$

Since U_i varies only slowly with z, the largest contribution will come from To and the streamfunction $u = \phi_z$, $w = -\phi_x$, $q = \nabla^2\phi$

$$
\begin{aligned}
\frac{\partial}{\partial t}\mathbf{u} + (\zeta + f)\hat{\mathbf{z}} \times \mathbf{u} = -\nabla(p + \frac{1}{2}\mathbf{u}^2) + \frac{\partial}{\partial z}\nu\frac{\partial}{\partial z}\mathbf{u} \\
\frac{\partial}{\partial z}p = b \\
\frac{\partial}{\partial t}b + \mathbf{u}\cdot\nabla b + w\frac{\partial}{\partial z}b = \frac{\partial}{\partial z}\kappa\frac{\partial}{\partial z}b \\
\nabla\cdot\mathbf{u} + \frac{\partial}{\partial z}w = 0
\end{aligned}
$$

Interior:
u, w small, non-diffusive

We shall consider the effects of a wind blowing along-shore. As Ekman realized, the transport in the near surface is 90° to the right of the wind (in the northern hemisphere). Essentially, on scales comparable to the mixed-layer depth, the pressure forces and advection of momentum are small, so that the force applied by the wind is balanced by Coriolis forces associated with the offshore flow. A southward wind on a western coast (Oregon, California...) gives off-shore transport in the surface layer, which must be compensated for by deeper on-shore flows (figure 1). The resulting flow begins to lift the density surfaces near the coast; in turn, the isopycnals tend to slump back towards level and begin to counteract the offshore tendency from the wind.

Figure : Sketch of upwelling system, showing along-shore wind, onshore deep flows, offshore surface flows, and upwelling near the coast.

As a first model of the flow, we shall make a number of simplifications (some of which will be remedied as we build a numerical model for the flow):

1) Straight coast: the topography, wind, and all flow variables are independent of the alongshore distance ($\frac{\partial}{\partial y} = 0$, c.f. Allen, 19xx). This kind of idealization is very useful, since it reduces a three dimensional problem to a two dimensional one. Yet it can be misleading; in the presence of strong alongshore currents, even small amplitude or large-scale downstream variations may be significant (Chapman, 19xx)
2) Weak flows: we take advection to be much smaller than the Coriolis, pressure gradient, and viscous terms. The Coriolis term will be written using only the vertical component of the rotation $2\vec{\Omega} = f\hat{\mathbf{z}}$. In the buoyancy equation, we assume that the deviations $B'(\mathbf{x},t) = B(\mathbf{x},t) - \overline{B}(z)$ from a stratified state are small and neglect advection of B' but not of $\overline{B}$. Likewise we split ϕ into $\overline{\phi} + \phi'$.
3) Since we cannot resolve the scales where molecular viscosity is a main contributor to the momentum balance (the Ekman scale, defined below, is about 10 cm), we take the common approach of regarding the back and forth exchange of momentum across a surface as caused by turbulence and having a larger effective viscosity.

Scale analysis

We can use the approach of scale analysis to decide the conditions under which such approximations might be valid. We define "scales" for the different variables: for the dependent variables, the scale value U, for example, would be the characteristic magnitude of u; for independent variables, the scales represent the characteristic magnitude of some field divided by the characteristic magnitude of its derivative so that we replace $\frac{\partial}{\partial x}$ by $1/L$ in estimating sizes of the different terms in an equation. We can then decide which terms are small compared to the others and drop them from subsequent analysis.

For example, mass equation and the scales of terms looks like

$$\begin{array}{ccc} \frac{\partial}{\partial x}u & + & \frac{\partial}{\partial z}w = 0 \\ \frac{U}{L} & & \frac{W}{H_\infty} \end{array}$$

where the horizontal and vertical scales are set by the topographic profile

$$H = H_\infty \tanh(x/L)$$

The last equation makes it clear that the use of mathematical functions necessitates specifying scales: we cannot define the hyperbolic tangent of a dimensional length such as 1.5 m. Trying to say it is tanh(1.5) as defined by a table or calculator does not work; would one then say that tanh of 1500 cm is tanh(1500)? † Rather, we always deal with functions acting on non-dimensional numbers which are the ratios of dimensional variables such as x to scales with the same dimensions, L. But extending the argument a little further, we could also be dealing with a shelf-slope topography $H = 10\ m \tanh(x/3\ km) + 500\ m \tanh([x - 150\ km]/50\ km) + 500\ m$ which has multiple scales for depth and length; therefore, scale analysis serves as a guide, but results should be checked *a-posteriori* to verify that neglected terms are indeed unimportant.

For the mass equation, however, we will not drop terms since that would leave us with an equation such as $\frac{\partial u}{\partial x} = 0$ which has only trivial solutions; instead, we use the scale analysis to find the sizes of terms for which we may not have an estimate for (such as W). Thus, we expect $W = UH/L$.

The momentum equations

$$\begin{array}{cccccccc}
\frac{\partial}{\partial t}u & + u\frac{\partial}{\partial x}u & + w\frac{\partial}{\partial z}u & - fv & = -\frac{\partial}{\partial x}\phi & + \nu\frac{\partial^2 u}{\partial x^2} & + \nu\frac{\partial^2 u}{\partial z^2} \\
\frac{U}{T} & \frac{U^2}{L} & \frac{WU}{H_\infty} & fV & \frac{\Phi}{L} & \frac{\nu U}{L^2} & \frac{\nu U}{H_\infty^2}
\end{array}$$

$$\begin{array}{cccccc}
\frac{\partial}{\partial t}v & + u\frac{\partial}{\partial x}v & + w\frac{\partial}{\partial z}v & + fu & = \nu\frac{\partial^2 v}{\partial x^2} & + \nu\frac{\partial^2 v}{\partial z^2} \\
\frac{V}{T} & \frac{UV}{L} & \frac{WV}{H_\infty} & fU & \frac{\nu V}{L^2} & \frac{\nu V}{H_\infty^2}
\end{array}$$

$$\begin{array}{ccccccc}
\frac{\partial}{\partial t}w & + u\frac{\partial}{\partial x}w & + w\frac{\partial}{\partial z}w & = -\frac{\partial}{\partial z}\phi & + B' & + \nu\frac{\partial^2 w}{\partial x^2} & + \nu\frac{\partial^2 w}{\partial z^2} \\
\frac{W}{T} & \frac{UW}{L} & \frac{W^2}{H_\infty} & \frac{\Phi}{H} & B' & \frac{\nu W}{L^2} & \frac{\nu W}{H_\infty^2}
\end{array}$$

suggest choosing $\Phi \sim fVL$, $U \sim \frac{\nu}{fH_\infty^2}V$, $B' \sim fVL/H_\infty$, $T \sim L/U$. Using these and normalizing the equations by the Coriolis or pressure terms gives the relative sizes:

$$\begin{array}{ccccc}
\frac{D}{Dt}u & - fv & = -\frac{\partial}{\partial x}\phi & + \nu\frac{\partial^2 u}{\partial x^2} & + \nu\frac{\partial^2 u}{\partial z^2} \\
\epsilon E^2 & 1 & 1 & \delta^2 E^2 & E^2
\end{array}$$

$$\begin{array}{cccc}
\frac{D}{Dt}v & + fu & = \nu\frac{\partial^2 v}{\partial x^2} & + \nu\frac{\partial^2 v}{\partial z^2} \\
\epsilon & 1 & \delta^2 & 1
\end{array}$$

$$\begin{array}{ccccc}
\frac{D}{Dt}w & = -\frac{\partial}{\partial z}\phi & + B' & + \nu\frac{\partial^2 w}{\partial x^2} & + \nu\frac{\partial^2 w}{\partial z^2} \\
\delta^2 \epsilon E^2 & 1 & 1 & \delta^4 E^2 & \delta^2 E^2
\end{array}$$

† To view the problem another way, note that we often define functions as a series: $\tanh(x) = x - \frac{x^3}{3} + \frac{2x^5}{15} \ldots$. If we tried to substitute 1.5 m into this formula, we'd have to add 1.5 m, 1.125 m^3, and 1.0125 m^5, which does not make physical sense.

where the non-dimensional parameters characterizing the flow are

$$\begin{aligned} \text{Rossby number} \quad & \epsilon = \frac{V}{fL} \\ \text{Aspect ratio} \quad & \delta = \frac{H_\infty}{L} \\ \text{Ekman number} \quad & E = \frac{\nu}{fH_\infty^2} \end{aligned}$$

These represent, respectively, the strength of advection (compared to Coriolis accelerations), the geometric constraint (which reduces the ratio of vertical to horizontal velocities), and the strength of friction (again compared to the Coriolis term). For the model we are now considering, we assume that all of these are small. Note that we still need to relate V to the external forcing; however, we can reduce the wind forcing until the Rossby number is indeed smaller than 1.

Over most of the flow, two important balances hold:

- The Coriolis force associated with the alongshore current is compensated by the cross-shelf pressure gradient; the near-equality of these forces is called **geostrophic balance** and applies in many larger scale flows, where it takes the more general form

$$f\hat{\mathbf{z}} \times \mathbf{u} = -\nabla\phi' \quad \Rightarrow \quad \mathbf{u} = \frac{1}{f}\hat{\mathbf{z}} \times \nabla\phi'$$

  In ocean eddies, as in atmospheric weather systems, the fluid moves along the lines of constant pressure, rather than accelerating down the gradient. In the northern hemisphere, the flow will have the high pressure to the right, and the speed will be proportional to the gradient (i.e., inversely proportional to the spacing between pressure contours – see figure 16).

- Vertically, the fluid remains in **hydrostatic balance**

$$\frac{\partial}{\partial z}\phi' = B'$$

  so that we can find the pressure by integrating the density field. However, this process introduces an unknown function of x, y, and t at the depth where the integration begins.

Now we examine the buoyancy equation

$$\begin{array}{cccc} \dfrac{D}{Dt}B' + & wN^2 = & \kappa\dfrac{\partial^2}{\partial x^2}B' + & \kappa\dfrac{\partial^2}{\partial z^2}B' \\ \dfrac{UB'}{L} & WN^2 & \dfrac{\kappa B'}{L^2} & \dfrac{\kappa B'}{H_\infty^2} \\ \epsilon & S & \dfrac{\delta^2}{Pr} & \dfrac{1}{Pr} \end{array}$$

where $N^2 \equiv \frac{\partial}{\partial z}\overline{B}$ is the square of the Brunt-Väisälä frequency. I.e., if we lift a blob of fluid in a stratified system, it is heavier than its surroundings (negatively buoyant) and accelerates downwards. It passes the initial position, decelerates, and comes to rest below, where it now feels a positive buoyancy force. The period of the resulting oscillation is $2\pi/N$.

The additional parameters above are

$$\begin{aligned} \text{Stratification} \qquad S &= \frac{N^2 H_\infty^2}{f^2 L^2} \\ \text{Prandtl number} \qquad Pr &= \frac{\nu}{\kappa} \end{aligned}$$

and are both assumed to be order one.

Thus we arrive at a simplified dynamics holding over most of the fluid:

$$\begin{aligned} fv &= \frac{\partial}{\partial x}\phi' \\ B' &= \frac{\partial}{\partial z}\phi' \\ \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} &= 0 \\ fu &= \nu \frac{\partial^2 v}{\partial z^2} \\ N^2 w &= \kappa \frac{\partial^2 B'}{\partial z^2} \end{aligned} \tag{1}$$

We can eliminate variables from these equations to get a single PDE for ϕ or for w; however, it is most convenient to work in terms of a streamfunction ψ for the flow.

Streamfunctions

For a two-dimensional incompressible flow, we can define the streamfunction difference between two points $\psi(\mathbf{x}_2) - \psi(x_1)$ as the volume of fluid passing through a surface formed by a curve joining the two points and sweeping a unit distance in the third direction. Because the flow in non-divergent, we will get the same answer for any curve joining the two points as long as it can be deformed into the original curve without crossing any obstacles in the flow. From this definition, we have

$$\begin{aligned} \psi(\mathbf{x}_2 + \delta x \hat{\mathbf{x}}) - \psi(\mathbf{x}_2) &= \delta x \frac{\partial \psi}{\partial x} = -w(\mathbf{x}_2)\delta x \\ \psi(\mathbf{x}_2 + \delta z \hat{\mathbf{z}}) - \psi(\mathbf{x}_2) &= \delta z \frac{\partial \psi}{\partial z} = u(\mathbf{x}_2)\delta z \end{aligned}$$

so that

$$u = \frac{\partial \psi}{\partial z} \quad , \quad w = -\frac{\partial \psi}{\partial x} \quad \mathit{or} \quad \mathbf{u} = -\nabla \times \psi \hat{\mathbf{y}}$$

which clearly satisfies the mass equation $\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$. The streamfunction acts much like the pressure in a geostrophic flow: instantaneously, the velocity vector is tangent to the contours of ψ, satisfying a "left-hand rule" (because of the negative sign), and the speed is inversely proportional to the separation of the ψ contours.

The upwelling equations can be reduced to an equation for ψ by first eliminating ϕ from the geostrophic and hydrostatic equations

$$f\frac{\partial v}{\partial z} = \frac{\partial B'}{\partial x}$$

(known as the thermal wind equation). The last two equations become

$$f\frac{\partial \psi}{\partial z} = \frac{\nu}{f}\frac{\partial^2 B'}{\partial x \partial z} \quad , \qquad -N^2\frac{\partial \psi}{\partial x} = \kappa\frac{\partial^2 B'}{\partial z^2}$$

from which we can eliminate the buoyancy to derive

$$\frac{f^2}{\nu}\frac{\partial^2 \psi}{\partial z^2} + \frac{N^2}{\kappa}\frac{\partial^2 \psi}{\partial x^2} = 0 \tag{2}$$

Boundary conditions

Now we must consider the boundary conditions to apply when solving eqn. 13. The flow normal to the bottom must be zero; this implies ψ should be constant ($\psi = 0$ since we can add an arbitrary constant to the streamfunction without affecting the flow) along the bottom. We might think that the same argument applies at the top, and the constant must be the same since $\int_{-H}^{0} dz u(x,z) = 0 = \psi(x,0) - \psi(x,-H)$; however, ψ would then be zero everywhere. What has gone wrong? The problem is that the full boundary conditions at the top are

$$w = 0 \quad , \quad \nu\frac{\partial v}{\partial z} = \frac{\tau}{\rho_s} \quad , \quad \nu\frac{\partial u}{\partial z} = 0 \quad at \quad z = 0$$

where τ is the wind stress and ρ_s the surface density. Our simplified system, eqns 11, cannot satisfy all of these simultaneously. There is a thin region near the surface with a characteristic scale $h_{ek} = \sqrt{\nu/f}$ which has an associated Ekman number of 1; in this region (the Ekman layer) other terms appear in the dynamics and permit us to match all of the conditions.

We can get at this directly by dropping only the terms which are order δ^2 or ϵ and then forming a streamfunction equation. The result

$$\frac{f^2}{\nu}\frac{\partial^2 \psi}{\partial z^2} + \frac{N^2}{\kappa}\frac{\partial^2 \psi}{\partial x^2} + \nu\frac{\partial^6 \psi}{\partial z^6} = 0$$

allows specifying three boundary conditions at the top and at the bottom (the no-normal flow plus two stress conditions at the top, and vanishing normal and tangential flow at the bottom).

However, we can gain more insight by examining the boundary layer flow directly; we define the velocities near the surface (shallower than $-h_{ek}$) to be those predicted by upwelling equations plus a correction $\mathbf{u}_{ek}$. These velocities vanish at depths $z << h_{ek}$. Since the upwelling equations define the velocities in terms of the pressure, we do not need a correction to ϕ. (This is related to the fact that solving the diagnostic equation for pressure will smooth out small scale structure in the velocities.) The correction equations

$$-fv_{ek} = \nu \frac{\partial^2}{\partial z^2} u_{ek}$$
$$fu_{ek} = \nu \frac{\partial^2}{\partial z^2} v_{ek}$$
$$\frac{\partial u_{ek}}{\partial x} + \frac{\partial v_{ek}}{\partial y} + \frac{\partial w_{ek}}{\partial z} = 0$$

can be integrated vertically to see that the horizontal transports in the surface layer are related to the wind stress

$$f \int v_{ek} = -\tau^{(x)}/\rho_s$$
$$f \int u_{ek} = \tau^{(y)}/\rho_s$$
$$\frac{\partial}{\partial x} \int u_{ek} + \frac{\partial}{\partial y} \int v_{ek} + w_{ek}(0) = 0$$

These take care of the two stress conditions; satisfying the no normal flow condition gives

$$w_{ek}(0) + w(0) = 0 \quad \Rightarrow \quad w(0) = \frac{\partial}{\partial x} \frac{\tau^{(y)}}{f\rho_s} - \frac{\partial}{\partial y} \frac{\tau^{(x)}}{f\rho_s}$$

For the upwelling problem, we have an offshore transport in the surface layer (for $\tau = \tau^{(y)} > 0$), and, if the wind increases offshore, that transport also increases. To provide this additional fluid, water must upwell from below. The effective boundary condition for the interior flow is

$$w = -\frac{\partial \psi}{\partial x} = \frac{\partial}{\partial x} \frac{\tau}{f\rho_s} \quad or \quad \psi = -\frac{\tau}{f\rho_s}$$

We take

$$\tau = \tau_0(1 - \exp(-x/L_\tau)$$

to avoid a singularity at the coast. To summarize, we must solve

$$\frac{\partial^2 \psi}{\partial z^2} + \frac{N^2 \nu}{f^2 \kappa} \frac{\partial^2 \psi}{\partial x^2} = 0$$
$$\psi = 0 \quad at \quad z = -H_\infty \tanh(x/L) \tag{3}$$
$$\psi = -\frac{\tau_0}{f\rho_s}(1 - \exp(-x/L_\tau) \quad at \quad z = 0$$

The solution is determined by two parameters $N^2 H_\infty^2 \nu / f^2 L^2 \kappa = S\ Pr$ and L_τ / L. The idealized flow used previously corresponds to the weakly stratified solution when $S\,Pr \ll 1$, so that equation (13) can be approximated by

$$\frac{\partial^2 \psi}{\partial z^2} \simeq 0$$

The streamfunction just linearly interpolates between the value at the base of the surface layer and the zero value at $z = -H_\infty \tanh(x/L)$. As we increase the strength of the stratification, the streamfunction spreads out horizontally (figure 17): the stronger stratification inhibits the vertical motion. The x scale where horizontal and vertical derivative are comparable is $R_d \sim NH/f$; it gives a rough measure of the width over which w will be significant.

Figure : Contours of ψ for $S\,Pr = 4 \times 10^{-2}$ (standard case) and $S\,Pr = 4$