Coastal plumes

Layer model

We shall work with a hydrostatic model, even though that can have issues. We consider a shallow, buoyant layer over a deep ocean (a $1\frac{1}{2}$ layer model). If the upper layer has density ρ_1 and extends from $z = \eta(\mathbf{x}, t)$ to $z = \eta(\mathbf{x}, t) - h(\mathbf{x}, t)$ and a lower layer of density ρ_2, the pressure at some depth H will be

$$p = \rho_1 g h + \rho_2 g (H - h + \eta)$$

If the deep layer is at rest $\nabla p = 0$, then

$$\rho_2 g \nabla \eta = (\rho_2 - \rho_1) g \nabla h$$

or

$$g \nabla \eta = g' \nabla h$$

with the reduced gravity being

$$g' = \frac{\rho_2 - \rho_1}{\rho_2} g$$

But the pressure gradient in the buoyant layer is just $\frac{1}{\rho_0} \nabla p = g \nabla \eta = g' \nabla h$. So the shallow water equations become

$$\frac{\partial}{\partial t} \mathbf{u} + (f + \zeta) \hat{\mathbf{z}} \times \mathbf{u} = -\nabla \left[g' h + \frac{1}{2} |\mathbf{u}|^2 \right]$$
$$\frac{\partial}{\partial t} h + \nabla \cdot (\mathbf{u} h) = 0$$

Outflow

As the buoyant water flows into the domain, there is no pressure gradient across it, so that the Coriolis force turns it anticyclonically. We could think of the jet as going in a semicircle leading back to the wall. So let's explore whether there is a solution for a jet travelling in a circle with $h = 0$ on either side. In polar coordinates, the azimuthal velocity $v(r)$ satisfies cyclo-geostrophic balance

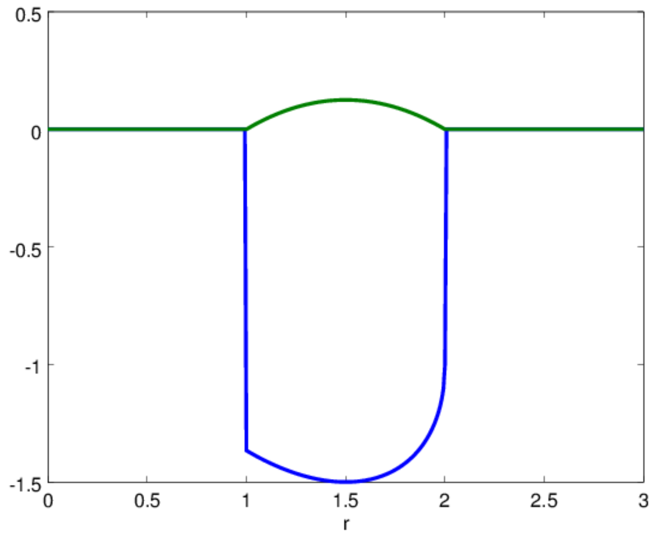
$$f v + \frac{v^2}{r} = \frac{\partial}{\partial r} g' h$$

If we integrate from the center of the circle outward, we want

$$\int dr \, \frac{v^2}{r} = -f \int dr \, v$$

The transport must be anticyclonic $\int v < 0$.

At this point, we can make up a structure of either v or h and solve for the other. In the former case, we may find that h goes negative, which we correct by cutting the domain to use just the annulus where $h \geq 0$. In the latter case, we have to choose the amplitude of h such that $rf^2 \geq 4\frac{\partial g' h}{\partial r}$. Both will end up with v profiles that have a jump.



Profiles of $v(r)$ [blue] and $g'h$ [green] with the latter chosen to be $\frac{1}{2}(2-r)(r-1)$ and $f = 1$.

This structure is likely to be unstable, and, furthermore, we would expect a lot of mixing with the deep fluid in the inner region where there is a strong shear layer.

Coastal plume

This current will run back into the coastand then proceed down with the coast on its right (NH) (Fong and Geyer, 2002, JPO 32, 957-72). The figure shows 3 cases: high inflow velocity, lower velocity, wide river.

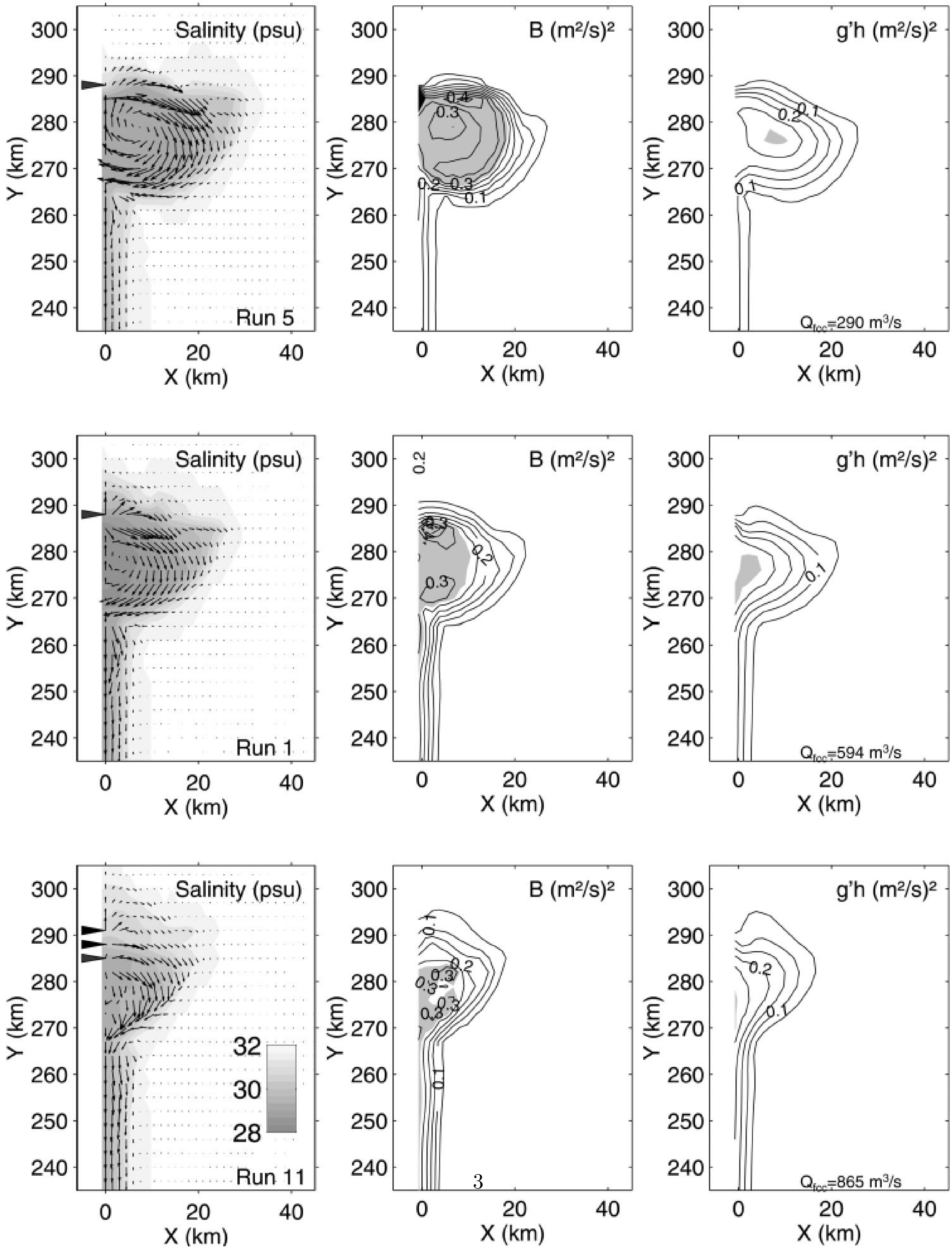


FIG. 6. Bulge behavior for different inflow conditions. The surface salinity and velocity, Bernoulli function

We can consider a 1.5 layer shallow water system

$$\frac{\partial}{\partial t}\mathbf{u} + (\zeta + f)\hat{\mathbf{z}} \times \mathbf{u} = -\nabla(g'h + \frac{1}{2}|\mathbf{u}|^2)$$
$$\frac{\partial}{\partial t}h + \nabla \cdot (\mathbf{u}h) = 0$$

This has a vorticity equation

$$\frac{\partial}{\partial t}(\zeta + f) + \nabla \cdot [\mathbf{u}(\zeta + f)] = 0$$

which can also be written in terms of the PV $q = (\zeta + f)/h$

$$\frac{\partial}{\partial t}hq + \nabla \cdot [\mathbf{u}hq] = 0$$

When combined with the mass equation, this gives PV conservation

$$\frac{\partial}{\partial t}q + \mathbf{u} \cdot \nabla q = 0$$

We consider a constant PV plume, $q = f/H$ (Stern, Whitehead, and Hua, 1982, JFM, 123, 237-265)

$$v_x - u_y = f\left(\frac{h}{H} - 1\right)$$

Suppose the x-scale is much larger than the width of the plume; then $v << u$. The y-momentum equations just gives geostrophic balance

$$fu = -g'h_y$$

and constan PV gives

$$u_y = -f\left(\frac{h}{H} - 1\right) \quad \Rightarrow \quad u_{yy} = \frac{f^2}{g'H}u + \frac{1}{R^2}u$$

As an example, let's take

$$u = U(x,t)e^{-y/R} \quad \Rightarrow \quad h = H\left[1 + \frac{1}{Rf}Ue^{-y/R}\right]$$

Substituting this into the u momentum equation and evaluating at $y = 0$ where $v = 0$ gives

$$\frac{\partial}{\partial t}U + U\frac{\partial}{\partial x}U + Rf\frac{\partial}{\partial x}U = 0$$

Linearizing this gives a Kelvin wave travelling at

$$c = \overline{U} + Rf$$

However, this equation also shows that there are no steadily propagating nonlinear disturbances

$$\frac{\partial}{\partial x}\left[-cU + \frac{1}{2}U^2 + RfU\right] = 0$$

so that either $U = 0$ or $U = 2(c - Rf)$; neither has downstream variation. Note that solving the momentum eqn. for $v(x, y, t)$ using the u, h solutions is consistent with the mass equation.

Stern et al. analyzed the finite width problem with $h = 0$ at $y = L(x, t)$.

$$h = H\left[1 - \cosh\frac{L-y}{R} + \frac{U}{Rf}\sinh\frac{L-y}{R}\right]$$

$$u = U\cosh\frac{L-y}{R} + Rf\sinh\frac{L-y}{R}$$

so that U is now the velocity along the front. Evaluating the momentum equation at $y = L$ and using the integrated mass

$$\frac{\partial}{\partial t}\int_0^L dy\, h/H + \frac{\partial}{\partial x}\int_0^L dy\, uh/H = 0$$

gives two equations for $U(x, t)$ and $L(x, t)$. As before, steady nonlinear waves do not exist; the waves will either steepen or flatten. Helfrich (2006) showed that a gravity current extends to the front and lengthens. It, too, can generate a nose where the semigeostrophic approx. breaks down (the two scales become comparable).

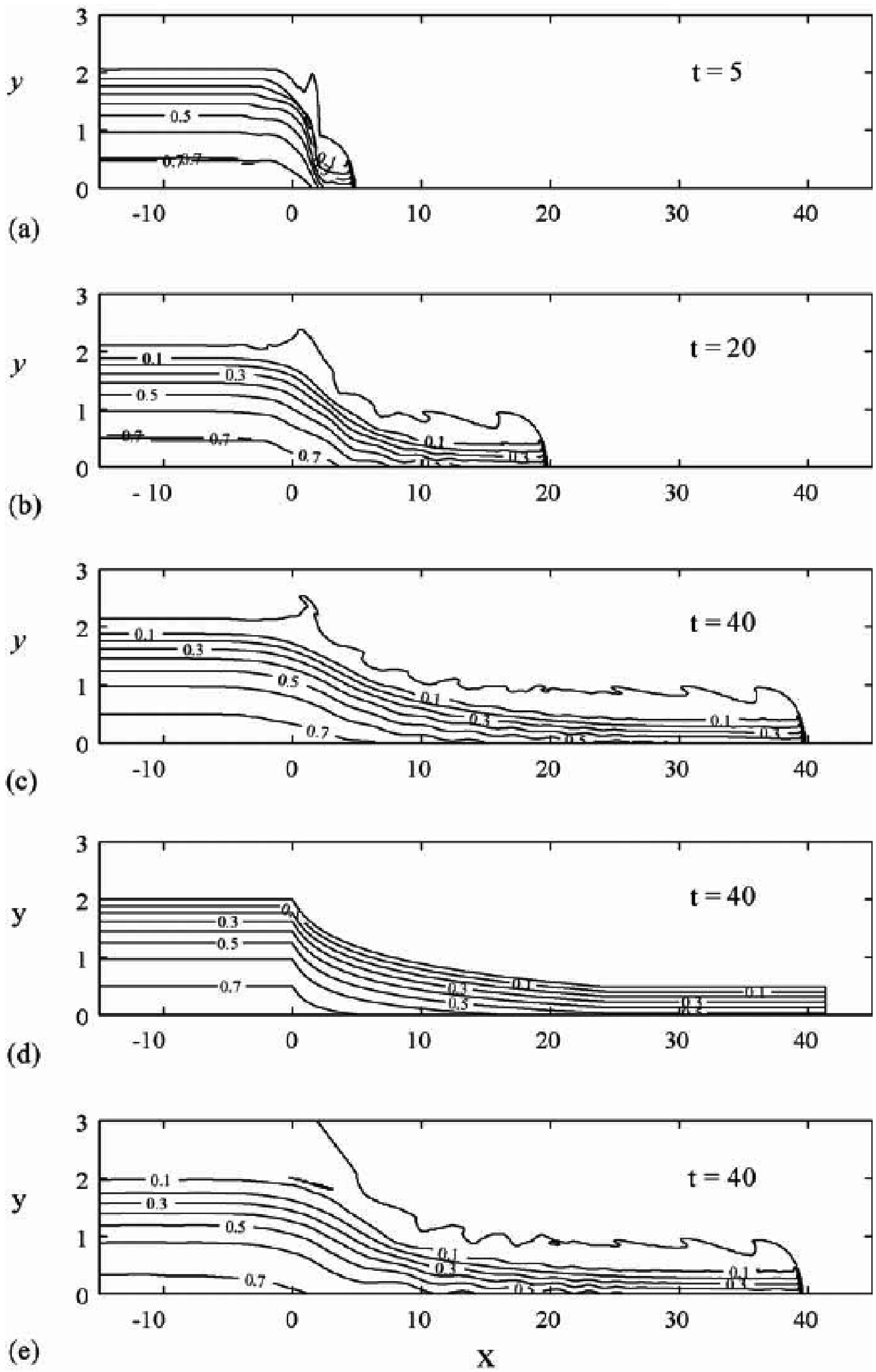


Fig. 10. Solutions for the downstream gravity current for $w_\infty = 2$. (a–c) The reduced-gravity numerical solution computed with the adjusted initial condition at $t = 5$, 20 and 40. (d) The semigeostrophic solution at $t = 40$. (e) Numerical solution at $t = 40$ computed with the still initial condition $w_0 = 1.03$. All panels show contours of $h(x, y, t)$.