Internal wave breaking

Convective breaking

- internal gravity wave in background stratification N^2 and zero mean flow

$$u = \frac{am}{K}\omega \cos\phi$$

$$v = + \frac{amf}{K} \sin\phi$$

$$w = -a\omega \cos\phi$$

$$b_T = N^2 z - N^2 a \sin\phi$$

with $\phi = Kx + mz - \omega t$

- monochromatic wave is an exact nonlinear solution

$$u u_x + w u_z = \frac{am}{K}\omega\left[-K\sin\phi\, u - m\sin\phi\, w\right] = \frac{am}{K}\sin\phi\left[-am\omega\cos\phi + am\omega\cos\phi\right] = 0$$

$$u v_x + w v_z = 0$$

$$u w_x + w w_z = 0$$

$$u b_x + w b_z = 0$$

- deflection of density surfaces

$$b_T = N^2(z - a\sin\phi)$$

show figure

$$\frac{db_T}{dz} = N^2(1 - \underbrace{am}_{\substack{\text{wave}\\\text{slope}\\ \| \\ S}}\cos\phi)$$

- $S \geq 1$ generates inversions unstable to convective instability

- regions of inversion is

$$1 - am\cos\phi \leq 0$$

$$\cos\phi \geq \frac{1}{am} = \frac{1}{S}$$

$$mh \geq 2\,\mathrm{acos}\,\frac{1}{S}$$

$$h \geq \frac{2a}{S}\,\mathrm{acos}\,\frac{1}{S}$$

- instability develops when

$$Ra = \frac{N^2 h^4}{\nu K} \gtrsim O(1000)$$

$$= \frac{h^2/\nu \;\; h^2/k}{1/N^2} = \frac{\text{viscous timescale} \times \text{diffusive timescale}}{(\text{buoyancy timescale})^2} \gtrsim O(1000)$$

- instability develops if

$$h^4 \gtrsim O(1000) \times \frac{\nu K}{N^2}$$

$$h \gtrsim \left[10^3 \times \frac{10^{-6}\,m^2/s \times 10^{-7}\,m^2/s}{10^{-4}\,s^{-2}} \right]^{1/4} = 10^{-6/4}\,m \simeq 3\,cm$$

- lee-waves

$$m = \sqrt{\frac{N^2}{U^2} - k^2} \leq \frac{N}{U}$$

$$a = h_0$$

$$\Rightarrow s = ma \leq h_0 \frac{N}{U} = O\left(10\text{-}100\,m \;\; \frac{10^{-3}\,m/s}{0.1\,m/s}\right) = O(10^{-1} - 1)$$

- tidal waves

CORRIGENDUM

m= k sqrt(N^2 -omega^2) / omega approx N k / omega
a U_0 h_0 k / omega
s=(U_0 k / omega)^2 (N h_0 / U_0)
s is much smaller than for lee waves, at least for low mode internal tides
which have small excursion parameters (U_0 k / omega)

show figures

much less than one except for ...

with scales $k = O(10^{-2})$ rad/m $\Rightarrow \lambda = \frac{2\pi}{10^{-2}}\,m \simeq 1\,km$

Shear breaking

- $Ri = \dfrac{b_{T,z}}{u_z^2}$ assuming $f = 0$

$$= \frac{N^2(1-s\cos\phi)}{\frac{a^2 m^4}{K^2} w^2 \sin^2\phi} = \frac{N^2(1-s\cos\phi)}{\frac{s^2 m^2}{K^2} N^2 \cos^2\theta \sin^2\phi} = \frac{1-s\cos\phi}{s^2\sin^2\phi} \frac{1}{\tan^2\theta\cos^2\theta}$$

- the minimum Ri for fixed s and θ is achieved for

$$\frac{\partial}{\partial\cos\phi}\left[\frac{1-s\cos\phi}{s^2(1-\cos^2\phi)}\right] = \frac{-s \times \cancel{s^2}(1-\cos^2\phi) - (1-s\cos\phi)\times(-2s^2\cos\phi)}{\left(s^2(1-\cos^2\phi)\right)^2} = 0$$

$$-s + s\cos^2\phi + 2\cos\phi - 2s\cos^2\phi = 0$$

$$s\cos^2\phi - 2\cos\phi + s = 0$$

$$\cos\phi = \frac{1 \pm \sqrt{1-s^2}}{s} = \frac{1-\sqrt{1-s^2}}{s}$$

$\hookrightarrow$ $\cos\phi$ must be smaller than 1
and s must be smaller than 1 (otherwise CI)

$$\sin^2\phi = 1 - \cos^2\phi = 1 - \frac{1+(1-s^2)-2\sqrt{1-s^2}}{s^2} = \frac{s^2-2+s^2+2\sqrt{1-s^2}}{s^2}$$

$$= 2\,\frac{s^2-1+\sqrt{1-s^2}}{s^2}$$

- $$Ri_{min} = \frac{1-s\cos\phi}{s^2\sin^2\phi}\frac{1}{\sin^2\theta} = \frac{1-\cancel{s}\frac{1}{\cancel{s}}(1-\sqrt{1-s^2})}{2\sqrt{1-s^2}(-\sqrt{1-s^2}+1)}\frac{1}{\sin^2\theta}$$

$$= \frac{1}{2}\frac{1}{1-\sqrt{1-s^2}}\frac{1}{\sin^2\theta} \geq \frac{1}{2}\frac{1}{1-\sqrt{1-s^2}} \geq \frac{1}{2}$$

flow is stable

• adding rotation

$$Ri = \frac{b_{T,z}}{u_z^2 + v_z^2} = \frac{N^2(1-s\cos\phi)}{\frac{a^2m^4}{K^2}\omega^2\sin^2\phi + \frac{a^2m^4}{K^2}f^2\cos^2\phi} = \frac{N^2(1-s\cos\phi)}{s^2\tan^2\theta\left(\omega^2\sin^2\phi + f^2\cos^2\phi\right)}$$

$$= \frac{1-s\cos\phi}{s^2\sin^2\phi}\,\frac{N^2}{\tan^2\theta\left[f^2\sin^2\theta + N^2\cos^2\theta + f^2\cot^2\phi\right]}$$

$$= \frac{1-s\cos\phi}{s^2\sin^2\phi}\,\frac{1}{\sin^2\theta}\,\frac{1}{1+\frac{f^2}{N^2}\tan^2\theta\left(\sin^2\theta + \cot^2\phi\right)}$$

for the term $\frac{f^2}{N^2}\tan^2\theta$ to become of $O(1)$ and reduce Ri_{min} the angle must be close to $\frac{\pi}{2}$

$\Rightarrow$ only near-inertial shear is unstable to K-H

Background flow

• in the presence of background flow

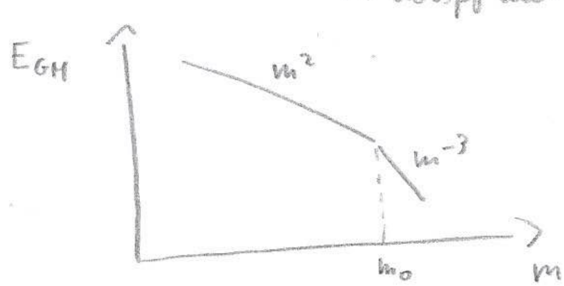
$$Ri = \frac{N^2 + b_z^2}{u_z^2 + U_z^2}$$

and the Ri can drop below critical if background flow is strong enough

• the GM spectrum for the shear predicts

$\langle u_z^2 + v_z^2 \rangle \simeq 0.8\ m^{+1} \times N^2 \times m_0 \simeq \frac{1}{2} N^2$ $\gg$ balanced shear

↳ cutoff wavenumber $\simeq 0.6\,m^{-1}$



• shorter waves must be important for wave breaking!

Weak turbulence

- how do IGW break, if generation results in stable waves?
- waves can interact nonlinearly
- waves satisfy the equations

$$\begin{cases} \partial_t \underset{\sim}{u} + f\hat{z}\times\underset{\sim}{u} + \nabla p - b\hat{z} = -(\underset{\sim}{u}\cdot\nabla)\underset{\sim}{u} \\ \partial_t b + N^2 w = -(\underset{\sim}{u}\cdot\nabla) b \\ \nabla\cdot\underset{\sim}{u} = 0 \end{cases}$$

- assuming that waves have small amplitude

$$(u, v, w, b, p) = O(\varepsilon)$$

at $O(\varepsilon)$

$$\begin{cases} \partial_t \underset{\sim}{u} + f\hat{z}\times\underset{\sim}{u} + \nabla p - b\hat{z} = 0 \\ \partial_t b + N^2 w = 0 \\ \nabla\cdot\underset{\sim}{u} = 0 \end{cases}$$

$$\begin{pmatrix} u \\ v \\ w \\ b \end{pmatrix} = \begin{pmatrix} (\cos\phi - i\frac{f}{\omega}\sin\theta)\sin\phi \\ (\sin\phi + i\frac{f}{\omega}\cos\theta)\sin\phi \\ -\cos\theta \\ -i\frac{N^2}{\omega}\cos\theta \end{pmatrix} a(\underset{\sim}{k}, t)\, e^{i\underset{\sim}{k}\cdot\underset{\sim}{x}}$$

$$\underset{\sim}{k} = k(\sin\theta\sin\phi, \sin\theta\cos\phi, \cos\theta)$$

$$\omega^2 = N^2\cos^2\theta + f^2\sin^2\theta$$

$$E = \frac{1}{2}\,\frac{1}{V}\iiint\left(u^2 + v^2 + w^2 + \frac{b^2}{N^2}\right)dx\,dy\,dz = a\,a^*$$

- linear equations can be written as

$$\partial_t a(\underset{\sim}{k}, t) + i\omega\, a(\underset{\sim}{k}, t) = 0$$

- adding nonlinear terms gives an equation of the form

$$\partial_t a(\underset{\sim}{k}_3, t) + i\omega_3\, a(\underset{\sim}{k}_3, t) = -\varepsilon i\omega_3 \sum_{\underset{\sim}{k}_1 + \underset{\sim}{k}_2 = \underset{\sim}{k}_3} \Gamma_{312}\, a(\underset{\sim}{k}_1, t)\, a(\underset{\sim}{k}_2, t) + \dots$$

where $a(\underset{\sim}{k}_i, t) = a(\underset{\sim}{k}_i)\, e^{-i\omega_i t}$

- if in addition to

$\underset{\sim}{k}_1 \pm \underset{\sim}{k}_2 = k_3$

we have

$\omega_1 \pm \omega_2 = \omega_3$

there is a resonance

$$\partial_t a(\underset{\sim}{k}_3, t) + i\omega_3 a(\underset{\sim}{k}_3, t) = -\varepsilon i \omega_3 \sum_{\underset{\sim}{k}_1 + \underset{\sim}{k}_2 = \underset{\sim}{k}_3} a(\underset{\sim}{k}_1)\, a(\underset{\sim}{k}_2)\, e^{-i(\omega_1 + \omega_2)t}$$

- this means that $a(\underset{\sim}{k}_3, t)$ must evolve on a slow time-scale

$$a(\underset{\sim}{k}_3, t) \equiv a(\underset{\sim}{k}_3, t, \underbrace{\varepsilon t}_{T})$$

to capture the fact that resonance implies $a(\underset{\sim}{k}_3, t)$ must change at leading order (albeit on a slow timescale)

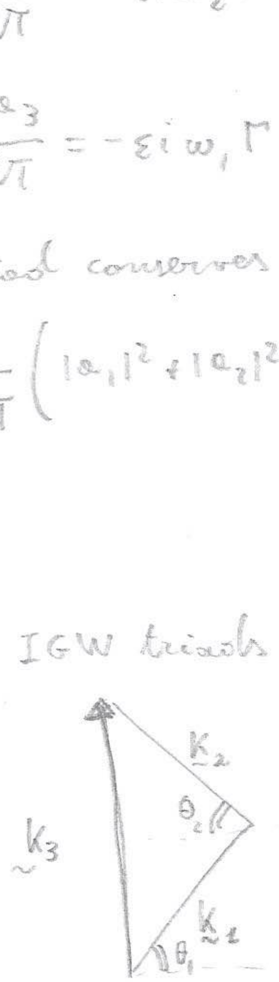
- consider a simple triad

$$\begin{cases} \dfrac{da_1}{dT} = -\varepsilon i \omega_1 \Gamma^* a_2^* a_3 \\[2ex] \dfrac{da_2}{dT} = -\varepsilon i \omega_2 \Gamma^* a_1^* a_3 \\[2ex] \dfrac{da_3}{dT} = -\varepsilon i \omega_1 \Gamma a_1 a_2 \end{cases}$$

- triad conserves energy

$$\begin{aligned} \frac{d}{dT}\left(|a_1|^2 + |a_2|^2 + |a_3|^2 \right) = &- \cancel{\varepsilon i \omega_1 \Gamma^* a_1^* a_2^* a_3} + \cancel{\varepsilon i \omega_1 \Gamma a_1 a_2 a_3^*} \\ &- \varepsilon i \omega_2 \Gamma^* a_1^* a_2^* a_3 + \varepsilon i \omega_2 \Gamma a_1 a_2 a_3^* \\ &- \cancel{\varepsilon i \omega_3 \Gamma a_1 a_2 a_3^*} + \cancel{\varepsilon i \omega_3 \Gamma^* a_1^* a_2^* a_3} \end{aligned}$$

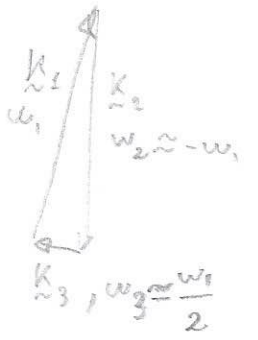
- do IGW triads exist?



$\theta_1 = \theta_2 \Rightarrow \omega_1 = \omega_2$ and $\omega_3 = 0$

Phillips (1966)

• PSI



transfers energy from primary high-frequency large-scale wave to low-frequency small-scale wave!

a route toward dissipation!