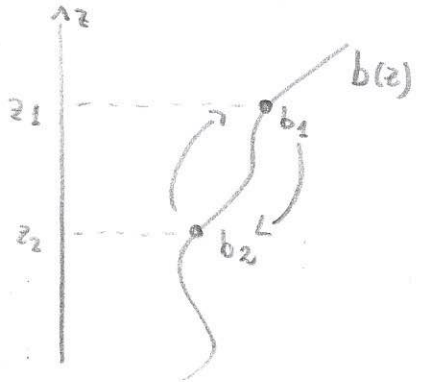
# Parcel theory of instabilities

- Gravitational instability

- suppose you have a fluid with variable density in the vertical (only PE in background)
- switch two particles with buoyancy $b_1$ and $b_2$




- $\Delta P = P_{final} - P_{initial}$

$$= -z_1 b_2 - z_2 b_1 - (-z_1 b_1 - z_2 b_2)$$

$$= (z_1 - z_2) b_1 - (z_1 - z_2) b_2$$

$$= \Delta z \Delta b = \Delta z^2 \frac{db}{dz} = \Delta z^2 N^2$$

- $\Delta P$ is released if $\Delta b < 0$, $b_1 - b_2 < 0$, $b_1 < b_2$ or $N^2 < 0$!

- Shear-instability

- fluid has variable $b(z)$ and $u(z)$
- two parcels have different densities

$$b_1 = b_2 + \frac{db}{dz} \Delta z$$

- two parcels have different velocities

$$u_1 = u_2 + \frac{du}{dz} \Delta z \qquad\xrightarrow[\text{after exchange}]{}\qquad u_2 = u_2 + \gamma \frac{du}{dz} \Delta z$$

$$u_2 = u_2 \qquad\qquad u_2 = u_2 + (1-\gamma) \frac{du}{dz} \Delta z$$

- change in PE

$$\Delta P = \Delta z \Delta b$$

- change in KE

$$\Delta K = \frac{1}{2}\left[-\left(u_2 + \frac{du}{dz}\Delta z\right)^2 - u_2^2 + \left(u_2 + \gamma\frac{du}{dz}\Delta z\right)^2 + \left(u_2 + (1-\gamma)\frac{du}{dz}\right)^2\right]$$

$$= \frac{1}{2}\left[\cancel{-2u_2\frac{du}{dz}\Delta z} - \left(\frac{du}{dz}\right)^2\Delta z^2 + \cancel{2\gamma u_2\frac{du}{dz}\Delta z} + \gamma^2\left(\frac{du}{dz}\right)^2\Delta z^2 \right.$$

$$\left. + \cancel{2(1-\gamma)u_2\frac{du}{dz}\Delta z} + (1-\gamma)^2\left(\frac{du}{dz}\right)^2\Delta z^2\right]$$

$$= +\frac{2}{2}\gamma(\gamma-1)\left(\frac{du}{dz}\right)^2\Delta z^2$$

- instability arises if

$$\Delta P + \Delta K \le 0$$

(instability always arises if fluid is not stratified)
(turbulence develops and erases any shear)

$$\frac{db}{dz}\Delta z^2 + \gamma(\gamma-1)\left(\frac{du}{dz}\right)^2\Delta z^2 \le 0$$

$$\Rightarrow \quad \frac{db}{dz} \le +\gamma(1-\gamma)\left(\frac{du}{dz}\right)^2 \le \frac{1}{4}\left(\frac{du}{dz}\right)^2 \qquad (\text{N.B. } 0<\gamma<1)$$

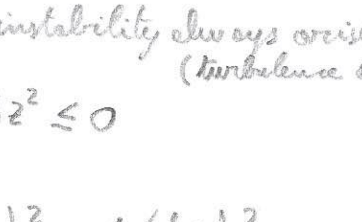
$$Ri = \frac{db/dz}{(du/dz)^2} \le \frac{1}{4}$$

- <u>Symmetric instability</u>

- consider a basic state with shear in geostrophic balance

$$f\frac{du}{dz} = -\frac{db}{dy} = -M^2$$

$$\frac{db}{dz} = N^2$$




$$b(y,z) = M^2 y + N^2 z$$

$$\Delta y = y_1 - y_2$$

$$\Delta z = z_1 - z_2$$

$$\Delta b = b_1 - b_2$$

- change in PE

$$\Delta P = \Delta z \Delta b = \Delta z(N^2\Delta z + M^2\Delta y)$$

introducing the slopes

$$s_b = -M^2/N^2, \qquad s = \Delta z/\Delta y \qquad \left[\text{N.B. } s_b = \frac{\Delta z_b}{\Delta y_b},\ db = M^2\Delta y_b + N^2\Delta z_b = 0\right]$$

$$\Delta P = N^2 \Delta y^2 s^2 - N^2 s_b^2 \Delta y \, s \Delta y = N^2 \Delta y^2 s (s - s_b)$$

- change in KE
  - without any x variations

$$\frac{Du}{Dt} - fv = \frac{D(u - fy)}{Dt} = 0$$

  - absolute momentum is conserved

$$\Delta u = f \Delta y$$

  - no variations in v and w in basic state to generate $\Delta v$ or $\Delta w$ by swapping

$$\Delta KE = \frac{1}{2}\left[ (u_1 - f\Delta y)^2 + (u_2 + f\Delta y)^2 - u_1^2 - u_2^2 \right]$$

$$= \frac{1}{2}\left[ -2u_1 f \Delta y + 2u_2 f \Delta y + 2f^2 \Delta y^2 \right]$$

$$= (u_2 - u_1) f \Delta y + f^2 \Delta y^2 = \Delta y^2 f \left( f - \frac{u_1 - u_2}{\Delta y} \right)$$

  - note that

$$u_1 - u_2 = \left( u_2 + \frac{du}{dy}\Delta y + \frac{du}{dz}\Delta z \right) - u_2 = \frac{du}{dy}\Delta y - \frac{M^2}{f}\Delta z$$

$$\Delta KE = \Delta y^2 f \left( f - \frac{du}{dy} + s\frac{M^2}{f} \right) = \Delta y^2 f \left( f - \frac{du}{dy} - \frac{N^2}{f} s s_b \right)$$

- the change in total energy

$$\Delta E = \Delta P + \Delta KE = \Delta y^2 \left[ f\left(f - \frac{du}{dy}\right) - N^2 s s_b + N^2 s (s - s_b) \right]$$

$$= \Delta y^2 \left[ f\left(f - \frac{du}{dy}\right) - N^2 s_b^2 + N^2 (s - s_b)^2 \right]$$

$$= \Delta y^2 \left[ f\left(f - \frac{du}{dy}\right) - \frac{f^2}{N^2}\left(\frac{du}{dz}\right)^2 + N^2 (s - s_b)^2 \right]$$

- minimum release of $\Delta E$ is along $s = s_b$

$$\Delta E_{min} = \Delta y^2 \left[ f\left(f - \frac{du}{dy}\right) - \frac{f^2}{N^2}\left(\frac{du}{dz}\right)^2 \right] \leq 0$$

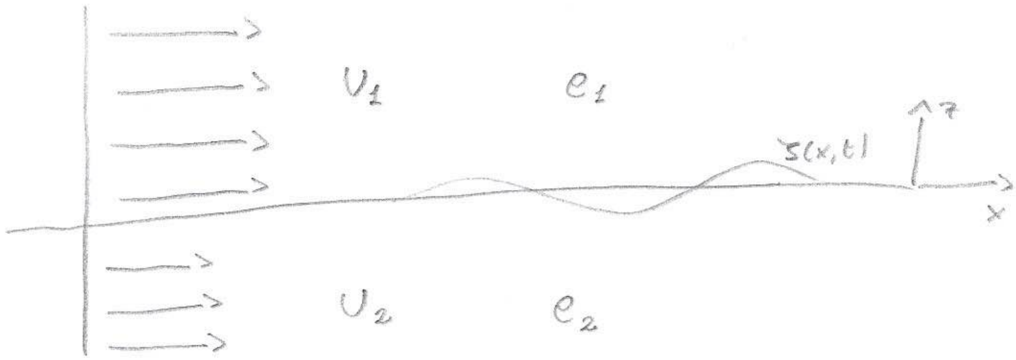
$$\Rightarrow f^2 \left( \frac{\zeta_{tot}}{f} - \frac{1}{Ri_b} \right) \leq 0 \quad \text{or} \quad Ri_b \leq \frac{f}{\zeta_{tot}} \quad \text{where } \zeta_{tot} = f - \frac{du}{dy}$$

- note that

$$Ri_b = \frac{N^2}{\left(\frac{du}{dz}\right)^2} = \frac{f^2 N^2}{M^4} \quad \text{the balanced } Ri$$

- relevant for waves in upper ocean where $Ri_b = O(1)$, otherwise $Ri_b \gg 1$ and symmetric instability is not likely to arise

# KH instability analysis




- flow is irrotational in both layers

$\tilde{\Phi}_1 = U_1 x + \phi_1 , \quad u_1 = \nabla \phi_1$

$\tilde{\Phi}_2 = U_2 x + \phi_2 , \quad u_2 = \nabla \phi_2$

- irrotational flow implies

$\nabla^2 \tilde{\Phi}_1 = \nabla^2 \phi_1 = 0$

$\nabla^2 \tilde{\Phi}_2 = \nabla^2 \phi_2 = 0$

- b.c. for $|z| \to \infty$

$\phi_1 \to 0$ as $z \to +\infty$

$\phi_2 \to 0$ as $z \to -\infty$

- interfacial condition (Kinematic b.c.)

$$w_1 = \frac{\partial \phi_1}{\partial z} = \frac{D\zeta}{Dt} = \frac{\partial \zeta}{\partial t} + U_1 \frac{\partial \zeta}{\partial x} \qquad @ z=0$$

$$w_2 = \frac{\partial \phi_2}{\partial z} = \frac{\partial \zeta}{\partial t} + U_2 \frac{\partial \zeta}{\partial x} \qquad @ z=0$$

- dynamic b.c.
  - momentum equations for an irrotational flow

$$\partial_t \nabla \tilde{\phi} + \frac{1}{2} \nabla (\nabla \tilde{\phi})^2 = -\frac{1}{\rho} \nabla \tilde{p} - \nabla g z$$

  - the two layers satisfy

$$\begin{cases} \partial_t \tilde{\phi}_1 + \frac{1}{2} (\nabla \tilde{\phi}_1)^2 + \frac{\tilde{p}_1}{\rho_1} + gz = C_1 \\ \partial_t \tilde{\phi}_2 + \frac{1}{2} (\nabla \tilde{\phi}_2)^2 + \frac{\tilde{p}_2}{\rho_2} + gz = C_2 \end{cases}$$

- the pressure is continuous at the interface

$\tilde{p}_1 = \tilde{p}_2$

- for basic state

$$\rho_1\left(\frac{1}{2}U_1^2 - C_1\right) = \rho_2\left(\frac{1}{2}U_2^2 - C_2\right)$$

- for perturbations

$$\rho_1\left(\partial_t \phi_1 + U_1 \partial_x \phi_1 + g\zeta\right) = \rho_2\left(\partial_t \phi_2 + U_2 \partial_x \phi_2 + g\zeta\right) \quad @ z=0$$

- writing solutions in the form

$$(\zeta, \phi_1, \phi_2) = (\hat{\zeta}(z), \hat{\phi}_1(z), \hat{\phi}_2(z))\, e^{ik(x-ct)}$$

$$\begin{cases} \hat{\phi}_{1,zz} - k^2\hat{\phi}_1 = 0, & \hat{\phi}_1 = Ae^{-kz} \\ \hat{\phi}_{2,zz} - k^2\hat{\phi}_2 = 0, & \hat{\phi}_2 = Be^{kz} \end{cases}$$

- Kinematic b.c.

$$(-ikc + ikU_1)\hat{\zeta} = \hat{\phi}_{1,z} = -Ak \quad \Rightarrow A = -i(U_1 - c)\hat{\zeta}$$

$$(-ikc + ikU_2)\hat{\zeta} = \hat{\phi}_{2,z} = Bk \quad \Rightarrow B = i(U_2 - c)\hat{\zeta}$$

- dynamic b.c.

$$\rho_1\left(-ikc\,\hat{\phi}_1 + ikU_1\hat{\phi}_1 + g\hat{\zeta}\right) = \rho_2\left(-ikc\,\hat{\phi}_2 + ikU_2\hat{\phi}_2 + g\hat{\zeta}\right) \quad @z=0$$

$$\rho_1\left(iK(U_1-c)(-i)(U_1-c)\hat{\zeta} + g\hat{\zeta}\right) = \rho_2\left(ik(U_2-c)\,i(U_2-c)\hat{\zeta} + g\hat{\zeta}\right)$$

$$\rho_1\left(k(U_1-c)^2 + g\right) = \rho_2\left(-k(U_2-c)^2 + g\right)$$

$$k\rho_1(U_1-c)^2 + k\rho_2(U_2-c)^2 = (\rho_2-\rho_1)g$$

$$k(\rho_1+\rho_2)c^2 - 2k(\rho_1U_1+\rho_2U_2)c + k\rho_1U_1^2 + k\rho_2U_2^2 - (\rho_2-\rho_1)g = 0$$

$$c = \frac{k(\rho_1U_1+\rho_2U_2) \pm \sqrt{k^2(\rho_1U_1+\rho_2U_2)^2 - k(\rho_1+\rho_2)\left(k\rho_1U_1^2 + k\rho_2U_2^2 - (\rho_2-\rho_1)g\right)}}{k(\rho_1+\rho_2)}$$

$$= \frac{\rho_1U_1+\rho_2U_2}{\rho_1+\rho_2} \pm \sqrt{\frac{g}{k}\frac{\rho_2-\rho_1}{\rho_2+\rho_1} - \rho_1\rho_2\left(\frac{U_1-U_2}{\rho_1+\rho_2}\right)^2}$$

- the flow is unstable if

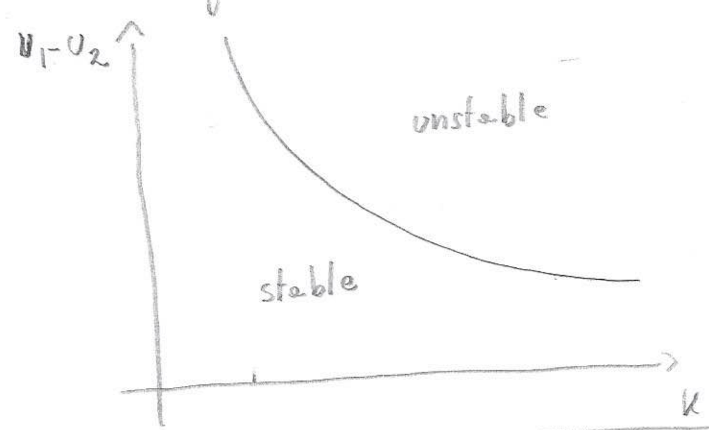
$$\frac{\cancel{g}}{k}\,\frac{\rho_2-\rho_1}{\cancel{\rho_2+\rho_1}} < \rho_1\rho_2\,\frac{(U_1-U_2)^2}{(\rho_1+\rho_2)^{\cancel{2}}}$$

$$g\,(\rho_2^2-\rho_1^2) < k\,\rho_1\rho_2\,(U_1-U_2)^2$$

- if $\rho_2=\rho_1$ flow is always unstable
- compare with KH criterion for weak density variations $\rho_2=\rho_1+\varepsilon$

$$g\,\frac{\Delta\rho}{\rho^2} < \frac{1}{2}k\,\Delta U^2$$

- instability curve




$$\omega_i = k c_i = k\sqrt{\rho_1\rho_2\left(\frac{U_1-U_2}{\rho_1+\rho_2}\right)^2-\frac{g}{k}\,\frac{\rho_2-\rho_1}{\rho_1+\rho_2}}$$

# Instability of continuously stratified parallel flows
## (Kelvin–Helmholtz instability)

- basic state $U(z)$
- perturbations

$[U+u, 0, w]$ Squire's theorem shows that 2D perturbations are most unstable

$P+p$

$B+b$

- perturbation equations

$$\begin{cases} \dfrac{\partial u}{\partial t} + U\dfrac{\partial u}{\partial x} + w\dfrac{dU}{dz} = -\dfrac{\partial p}{\partial x} \\ \dfrac{\partial w}{\partial t} + U\dfrac{\partial w}{\partial x} = -\dfrac{\partial p}{\partial z} + b \\ \dfrac{\partial u}{\partial x} + \dfrac{\partial w}{\partial z} = 0 \\ \dfrac{\partial b}{\partial t} + U\dfrac{\partial b}{\partial x} + w\underbrace{\dfrac{dB}{dz}}_{N^2} = 0 \end{cases}$$

- introducing the streamfunction

$u = -\psi_z$

$w = \psi_x$

- perturbation equations become

$$\begin{cases} -\psi_{zt} - U\psi_{xz} + \psi_x U_z = -p_x \\ \psi_{xt} + U\psi_{xx} = -p_z + b \\ b_t + Ub_x + N^2\psi_x = 0 \end{cases}$$

- ignoring lateral boundaries we can express solutions in the form

$[b, p, \psi] = [\hat{b}(z), \hat{p}(z), \hat{\varphi}(z)]\, e^{ik(x-ct)}$

where $k$ must be real for the flow to remain bounded

$c = c_r + ic_i$ can be complex ($c_i = 0$ waves, $c_i \neq 0$ unstable modes)

- perturbation equations

$$\begin{cases} (U-c)\,\hat{\psi}_z - U_z\,\hat{\psi} = +\hat{p} \\ (U-c)\,K^2\,\hat{\psi} = +\hat{p}_z - \hat{b} \\ (U-c)\,\hat{b} + N^2\,\hat{\psi} = 0 \end{cases}$$

- taking z-derivative of first equation and subtracting second equation

$$\cancel{U_z\hat{\psi}_z +} (U-c)(\partial_z^2 - K^2)\,\hat{\psi} - U_{zz}\,\hat{\psi} \cancel{- U_z\,\hat{\psi}_z} + \frac{N^2}{U-c}\,\hat{\psi} = 0$$

Taylor-Golstein equation

- boundary conditions of no-normal flow at top and bottom boundaries

$$\Psi(0) = \Psi(H) = 0$$

- introduce new variable

$$\phi = \frac{\hat{\psi}}{\sqrt{U-c}} \quad \text{or} \quad \hat{\psi} = (U-c)^{1/2}\,\phi$$

- T-G equation reduces to

$$\hat{\psi}_z = (U-c)^{1/2}\,\phi_z + \frac{1}{2}\frac{U_z}{(U-c)^{1/2}}\,\phi$$

$$\hat{\psi}_{zz} = (U-c)^{1/2}\,\phi_{zz} + \frac{U_z}{(U-c)^{1/2}}\,\phi_z + \frac{1}{2}\frac{U_{zz}}{(U-c)^{1/2}}\,\phi - \frac{1}{4}\frac{U_z^2}{(U-c)^{3/2}}\,\phi$$

$$(U-c)\left[(U-c)^{1/2}\,\phi_{zz} + \frac{U_z}{(U-c)^{1/2}}\,\phi_z + \frac{1}{2}\frac{U_{zz}}{(U-c)^{1/2}}\,\phi - \frac{1}{4}\frac{U_z^2}{(U-c)^{3/2}}\,\phi - K^2(U-c)^{1/2}\,\phi\right]$$

$$+ \left[\frac{N^2}{U-c} - U_{zz}\right](U-c)^{1/2}\,\phi = 0$$

$$\frac{d}{dz}\Big((U-c)\,\phi_z\Big) + \left(-\frac{1}{2}U_{zz} - \frac{1}{4}\frac{U_z^2}{(U-c)} - K^2(U-c) + \frac{N^2}{U-c}\right)\phi = 0$$

- multiplying by the complex conjugate $\phi^*$ and integrating in the vertical

- here it goes

$$\int_0^H \left[ (U-c)\left(-|\phi_z|^2\right) - K^2 (U-c)|\phi|^2 \right] dz +$$

$$+ \int_0^H \frac{N^2 - \frac{1}{4}U_z^2}{U-c} |\phi|^2 \, dz + \frac{1}{2}\int_0^H U_{zz} |\phi|^2 dz = 0$$

- the imaginary part of this expression is

$$c_i \int_0^H \left[ |\phi_z|^2 + K^2 |\phi|^2 \right] dz + c_i \int_0^H \frac{N^2 - \frac{1}{4}U_z^2}{|U-c|^2} |\phi|^2 dz = 0$$

- if $N^2 > \frac{1}{4}U_z^2$ then $c_i = 0$

- if $N^2 < \frac{1}{4}U_z^2$ then $c_i$ can be $\neq 0$

$Ri = \dfrac{N^2}{U_z^2} < \dfrac{1}{4}$ is a necessary (but not sufficient) condition for instability!

$Ri = \dfrac{N^2}{U_z^2} > \dfrac{1}{4}$ guarantees stability

- notice that Ri can fall below $\frac{1}{4}$ only in a limited region for instability to possibly develop