Convection

Convection is the first example we shall study of turbulence which is not homogeneous or isotropic, It draws on the potential energy which is available when the fluid is negatively stratified or has horizontal buoyancy variations. For the Boussinesq equations

$$\frac{D}{Dt}\mathbf{u} + f\hat{\mathbf{z}} \times \mathbf{u} = -\nabla p + b\hat{\mathbf{z}} + \nu\nabla^2\mathbf{u}$$

$$\nabla \cdot \mathbf{u} = 0$$

$$\frac{D}{Dt}b = \kappa\nabla^2 b$$

the kinetic energy satisfies

$$\frac{D}{Dt}\frac{1}{2}|u_i|^2 = -\nabla \cdot (\mathbf{u}p) - \nu\left|\frac{\partial u_i}{\partial x_j}\right|^2 + wb \qquad (KE)$$

The last term represents the conversion of available potential energy to kinetic energy. We can see the connection to potential energy by noting that

$$\frac{D}{Dt}(-zb) = -wb - z\kappa\nabla^2 b = -wb + \kappa\nabla^2(-zb) + 2\kappa\frac{\partial}{\partial z}b \qquad (PE)$$

so that wb is a sink for PE: raising lighter (buoyant) fluid and lowering heavier fluid decreases the potential energy. The definition $-zb$ corresponds to the standard mass$\times$gravity$\times$height for a unit volume, once we remember

$$\rho = \rho_0(1 - \frac{b}{g})$$

Note that we really have to compare the potential energy of the entire system to the energy it would have in some other obtainable state to decide if the transition is possible. In the figure on the following page, the various states have potential energies as follows:

state	PE	transition	δPE
a	$-27b_\ell - 21b_h$	$\to$ b	$-18(b_h - b_\ell)$
b	$-45b\ell - 3b_h$		
c	$-24b_\ell - 24b_h$	$\to$ d	$-12(b_h - b_\ell)$
d	$-36b_\ell - 12b_h$		
d'	$-12b_\ell - 24b_h$	$\to$ d	$-12(b_h - b_\ell)$
e	$-\frac{88}{3}b_\ell - \frac{56}{3}b_h$	$\to$ d	$-\frac{20}{3}(b_h - b_\ell)$
f	$-24(b_h + b_\ell)$	$\leftarrow$ d	$12(b_h - b_\ell)$

where d' is an inverted version of d. From this, we see that either horizontal gradients or inverted density profiles release potential energy when the heavy fluid settles to the bottom (no surprises here!). However, we also see that the potential energy of a mixed state is actually higher than that of a stably-stratified state. In the potential energy equation, this shows up as the $2\kappa\frac{\partial}{\partial z}b$ term – diffusion acting on a stable state increases the center of mass height and the potential energy. (The energy to do this must come from the molecular kinetic energy; the Boussinesq equations sweep this under the rug a bit in that internal energy is not well-represented.)

a   b

c   d

e   f
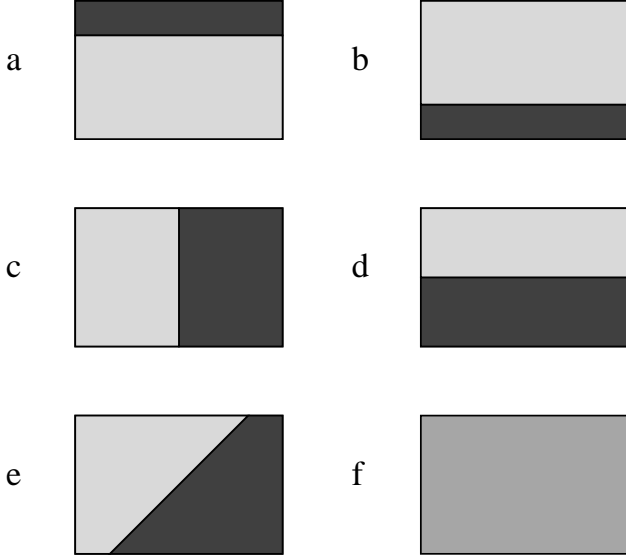
Various states with lighter fluid b_ℓ, heavier fluid $b_h < b_\ell$ and a mixture.

When dense fluid overlies less dense fluid, the layer will tend to overturn. To get any idea of when this might occur, consider a blob of fluid of size h rising with speed W through a fluid where the temperature decreases with height $\partial \overline{T}/\partial z < 0$. The temperature of the blob, T, will reflect the environment with some time lag for the outside temperature to diffuse in. On the rising particle, we can think of the heat budget as

$$\frac{d}{dt}T = \frac{\kappa}{h^2}\left[\overline{T}(Z(t)) - T\right] = \frac{\kappa}{h^2}\left[\frac{\partial \overline{T}}{\partial z}Z(t) - T\right]$$

The parcel feels a buoyancy force $\alpha g(T - \overline{T})$ with α being the thermal expansion coefficient $\rho = \rho_0 - \alpha(T - T_0)$; this force must balance or overcome the viscous drag $\nu W/h^2$ where ν is the viscosity.

$$\frac{d^2}{dt^2}Z = -\frac{\nu}{h^2}\frac{d}{dt}Z + \alpha g(T - \frac{\partial \overline{T}}{\partial z}Z)$$

We can look for $\exp(\sigma t)$ solutions to these in which case

$$\sigma^2 + \sigma\frac{\nu + \kappa}{h^2} + \alpha g\frac{\partial \overline{T}}{\partial z} + \frac{\nu\kappa}{h^4} = 0$$

and unstable solutions ($\sigma > 0$) will exist for

$$\alpha g\frac{\partial \overline{T}}{\partial z} + \frac{\nu\kappa}{h^4} < 0 \quad \Rightarrow \quad Ra \equiv \frac{-\alpha g\frac{\partial \overline{T}}{\partial z}h^4}{\nu\kappa} > 1$$

The neutral solutions $Ra = 1$ are simple: the fluid rises steadily

$$Z = Wt \quad \Rightarrow \quad T = \frac{\partial \overline{T}}{\partial z}Wt - W\frac{h^2}{\kappa}\frac{\partial \overline{T}}{\partial z} \quad \Rightarrow \quad b = \alpha g(T - \overline{T}) = -W\frac{h^2}{\kappa}\alpha g\frac{\partial \overline{T}}{\partial z}$$

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so that the temperature lags its surroundings by a fixed amount, and the buoyancy force is constant and can balance the drag

$$-W\frac{h^2}{\kappa}\alpha g\frac{\partial \overline{T}}{\partial z} = \frac{\nu}{h^2}W$$

Rayleigh-Benard

The classic Rayleigh-Benard problem begins with the dynamical equations for conservation of mass and momentum and buoyancy. We shall add rotation

$$\frac{D}{Dt}\mathbf{u} + f\hat{\mathbf{z}} \times \mathbf{u} = -\nabla P + b\,\hat{\mathbf{z}} + \nu\nabla^2\mathbf{u}$$

$$\nabla \cdot \mathbf{u} = 0$$

$$\frac{D}{Dt}b = \kappa\nabla^2 b$$

Note: There's no prognostic equation for P; how can we step the equations forward? The original equations can be written in terms of a prognostic system for $\mathbf{u}$, ρ, and p. In principle, it can be stepped forward, although, as L.F. Richardson found out, sound waves make any such attempt problematical at best. For the Boussinesq equations, you have to find P diagnostically: if you take the divergence of the momentum equations, the $\frac{\partial}{\partial t}$ term disappears and you're left with

$$\nabla^2 P = f\hat{\mathbf{z}} \cdot \nabla \times \mathbf{u} - \frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i} + \frac{\partial b}{\partial z}$$

from which you can compute P. If we use $\mathbf{u} = -\nabla\phi - \nabla \times \boldsymbol{\psi}$ with $\nabla \cdot \boldsymbol{\psi} = 0$, the first term is just $f\nabla^2\psi_3$ and relates the streamfunction to the pressure — they balance under geostrophic conditions.

Vector invariant form

The $(\mathbf{u} \cdot \nabla)\mathbf{u}$ is well-defined in Cartesian coordinates, but not in other systems; therefore, it is usefully to write the momentum equation in a form which can be converted to polar, spherical, ellipsoidal... coordinates by using standard forms (e.g., Morse and Feshbach)

$$\frac{\partial}{\partial t}\mathbf{u} + (f\hat{\mathbf{z}} + \boldsymbol{\zeta}) \times \mathbf{u} = -\nabla P - \nabla(\frac{1}{2}\mathbf{u} \cdot \mathbf{u}) + \hat{\mathbf{z}}b - \nu\nabla \times \boldsymbol{\zeta}$$

where $\boldsymbol{\zeta}$ is the vorticity $\nabla \times \mathbf{u}$. Note again the resemblance of the vorticity to twice the rotation rate. You can think of the parcel as accelerating because of gradients in the Bernoulli function $P + \frac{1}{2}\mathbf{u} \cdot \mathbf{u}$ and Coriolis forces associated with planetary and local rotation.

Vorticity equation

Taking the curl gives the equation for the time rate of change of the absolute vorticity $\mathbf{Z} = \boldsymbol{\zeta} + f\hat{\mathbf{z}}$

$$\frac{\partial}{\partial t}\mathbf{Z} + \nabla \times (\mathbf{Z} \times \mathbf{u}) = -\hat{\mathbf{z}} \times \nabla b - \nu\nabla \times (\nabla \times \mathbf{Z})$$

or (back in Cartesian form, using the non-divergence of $\mathbf{u}$ and $\mathbf{Z}$)

$$\frac{D}{Dt}\mathbf{Z} = (\mathbf{Z} \cdot \nabla)\mathbf{u} - \hat{\mathbf{z}} \times \nabla b + \nu\nabla^2\mathbf{Z}$$

Vorticity is generated by stretching and twisting or by buoyancy forces.

Non-dimensional form

We scale $\mathbf{x}$ by h, t by h^2/κ, $\mathbf{u}$ by $h/[t] = \kappa/h$, and buoyancy by $[w]|S|[t] = h|S|$ where S is the background stratification $\frac{\partial}{\partial z}\bar{b}$ (and is negative) set by the top and bottom boundary conditions. Finally, the scaling for the pressure is chosen to balance viscosity $\nu[w]/h = \nu\kappa/h^2$.

$$\frac{1}{Pr}\frac{D}{Dt}\mathbf{u} + T^{1/2}\hat{\mathbf{z}} \times \mathbf{u} = -\nabla P + Ra\,b\,\hat{\mathbf{z}} + \nabla^2\mathbf{u}$$

$$= -\nabla P' + Ra\,b'\,\hat{\mathbf{z}} + \nabla^2\mathbf{u}$$

$$\nabla \cdot \mathbf{u} = 0$$

$$\frac{D}{Dt}b = \nabla^2 b$$

or

$$\frac{D}{Dt}b' - w = \nabla^2 b'$$

with $b = -z + b'(\mathbf{x}, t)$.

The parameters are

$$Ra = \frac{|S|h^4}{\nu\kappa} \quad , \quad T = \frac{f^2h^4}{\nu^2} \quad , \quad Pr = \frac{\nu}{\kappa}$$

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Linear stability

The classical problem considers a fluid with an unstable stratification confined between horizontal plates at $z = 0, \ h$. The temperatures on the plates are held fixed, and the boundaries are assumed to be stress-free (this can be fixed, but makes the math messier – see Chandrasekhar). The linearized equations become

$$\frac{1}{Pr}\frac{\partial}{\partial t}\mathbf{u} + T^{1/2}\hat{\mathbf{z}} \times \mathbf{u} = -\nabla P + Ra\, b\, \hat{\mathbf{z}} + \nabla^2\mathbf{u}$$

$$\nabla \cdot \mathbf{u} = 0$$

$$\frac{\partial}{\partial t}b - w = \nabla^2 b$$

$$\frac{1}{Pr}\frac{\partial}{\partial t}\mathbf{Z} = T^{1/2}\frac{\partial w}{\partial z}\hat{\mathbf{z}} - Ra\,\hat{\mathbf{z}} \times \nabla b + \nabla^2\mathbf{Z}$$

We define two operators

$$D_\nu = \frac{1}{Pr}\frac{\partial}{\partial t} - \nabla^2 \quad , \quad D_\kappa = \frac{\partial}{\partial t} - \nabla^2$$

and start with the vertical vorticity equation

$$D_\nu \zeta = T^{1/2}\frac{\partial w}{\partial z}$$

and the Laplacian of the vertical momentum equation

$$D_\nu \nabla^2 w = -\frac{\partial}{\partial z}\nabla^2 P + Ra\nabla^2 b$$

Using the diagnostic equation for the pressure

$$\nabla^2 P = T^{1/2}\zeta + Ra\frac{\partial b}{\partial z}$$

gives

$$D_\nu \nabla^2 w = -T^{1/2}\frac{\partial \zeta}{\partial z} + Ra\nabla_h^2 b$$

with ∇_h^2 being the horizontal Laplacian. Eliminating the vertical vorticity yields

$$D_\nu^2 \nabla^2 w + T\frac{\partial^2 w}{\partial z^2} + Ra\nabla_h^2 D_\nu b$$

which will be combined with the buoyancy equation

$$D_\kappa b = w$$

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to get a single equation for w

$$D_\nu^2 D_\kappa \nabla^2 w + T D_\kappa \frac{\partial^2}{\partial z^2} w - Ra D_\nu \nabla_h^2 w = 0$$

For positive $S = \frac{\partial \bar{b}}{\partial z}$, $Ra < 0$, this gives damped internal waves; however, we're interested in the development for $S < 0$.

Graphics: planforms theta=0 theta=45,135 theta=0,60,120 p greater than -0.33 theta=0,55,118 theta=0,99,162,274 theta=0,72,144,216,288

At this point we make a horizontal planform statement

$$\nabla_h^2 w = -k^2 w$$

and a vertical structure form $w = \sin(\pi z)$

$$\frac{\partial^2 w}{\partial z^2} = -m^2 w \quad , \quad m = \pi$$

This is where boundary conditions come in: we have originally a sixth order equation for w so that we need three conditions on w, the obvious one being $w = 0$. The sinusoidal solution has the perturbation b zero, consistent with fixed temperatures on the plates. It also has $\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0$ corresponding to free slip.

Growth rates

We continue separating variables, now in time:

$$w = W(x, y) \sin(\pi z) e^{\sigma t}$$

so that $D_\kappa = \sigma + A$ with $A = k^2 + \pi^2$. The flow will be unstable when $Re(\sigma) > 0$; the growth/ propagation rates are the eigenvalues of the matrix and satisfy the characteristic equation

$$\left(\frac{1}{Pr}\sigma + A\right)^2 (\sigma + A) - Ra\, k^2 \frac{\sigma/Pr + A}{A} + T\pi^2 \frac{\sigma + A}{A} = 0$$

In the inviscid case, $\nu = \kappa = 0$, the growth rates (dimensional) are

$$\sigma = 0 \quad , \quad \pm\sqrt{-\frac{k^2 S + f^2 m^2}{k^2 + m^2}}$$

and instability requires

$$k^2 S + m^2 f^2 < 0$$

which will occur for sufficiently small scale motions whenever $S < 0$. The resting state is unstable when the density increases with height (or the buoyancy decreases).

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Details of growth rate eqn.

For convenience, we replace $\sigma = Pr\,\tilde{\sigma}$ (scaling time by the viscous rather than diffusive scale). Then we have

$$(\tilde{\sigma} + A)^2 (Pr\,\tilde{\sigma} + A) - Ra\,k^2 \frac{\tilde{\sigma} + A}{A} + T\pi^2 \frac{Pr\,\tilde{\sigma} + A}{A} = 0$$

Near the critical point $A^3 - Ra\,k^2 + T\pi^2 = 0$ we have

$$\tilde{\sigma} \simeq \frac{Ra\,k^2 - T\pi^2 - A^3}{A^2(2 + Pr) - (Ra\,k^2 - T\pi^2 Pr)/A^2}$$

$$\simeq \frac{Ra\,k^2 - T\pi^2 - A^3}{Pr\,A^2 + (Pr - 1)T\pi^2/A^2}$$

Graphics: growth rate surfaces T=0 T=100 T=1000
The growth rate σ will pass through zero on the real axis when

$$A^3 - Ra\,k^2 + \pi^2 T = 0$$

or

$$Ra = \frac{(k^2 + \pi^2)^3 + \pi^2 T}{k^2}$$

Graphics: Stability bndry various Ra T closeup closer
The smallest critical Rayleigh number occurs at $T = 0$, $k^2 = \frac{1}{2}\pi^2$ and is

$$Ra = \frac{27}{4}\pi^4 \simeq 658$$

corresponding to a temperature change over 1 meter of $4 \times 10^{-8}\ {}^\circ\mathrm{C}$ using $\alpha = 2.5 \times 10^{-4}/{}^\circ\mathrm{C}$, $\nu = 10^{-2}\ cm^2/s$, $\nu/\kappa = 7$. (The Taylor number with $f = 10^{-4}/s$ is about 1.)

There can be an instability in which $Re(\sigma)$ passes through zero at a point where $Im(\sigma) \neq 0$. For Prandtl number $\nu/\kappa = 7$, however, this happens at a Rayleigh number larger than the value above.

Nonlinear dynamics

2-D convection

When we consider convective motions which are independent of x, the zonal component of the vorticity equation gives us

$$\frac{1}{Pr}\frac{D}{Dt}\xi = T\frac{\partial}{\partial z}\tilde{u} + Ra\frac{\partial}{\partial y}b' + \nabla^2\xi \quad , \quad \nabla^2\phi = \xi$$

where ϕ is the streamfunction ($w = \frac{\partial\phi}{\partial y}$, $v = -\frac{\partial\phi}{\partial z}$ and ξ the x-component of $\mathbf{Z}$ ($= \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = \nabla^2\phi$). The other two equations are

$$\frac{D}{Dt}b' = \frac{\partial}{\partial y}\phi + \nabla^2 b'$$

$$\frac{1}{Pr}\frac{D}{Dt}\tilde{u} = -\frac{\partial}{\partial z}\phi + \nabla^2\tilde{u}$$

with $u = T^{1/2}\tilde{u}$.

The stability problem has constant coefficients so that

$$\xi = \xi_0\sin(ky)\sin(\pi z)e^{\sigma t}$$
$$b = b_0\cos(ky)\sin(\pi z)e^{\sigma t} \quad \text{(dropping primes)}$$
$$u = u_0\sin(ky)\cos(\pi z)e^{\sigma t} \quad \text{(dropping tildes)}$$
$$\phi = -\frac{\xi_0}{A}\sin(ky)\sin(\pi z)e^{\sigma t}$$
$$w = (\sigma + A)b$$

($A = k^2 + \pi^2$) and the equations become

$$\begin{pmatrix} -A & -\pi T & -k\,Ra \\ \frac{\pi}{A} & -A & 0 \\ -\frac{k}{A} & 0 & -A \end{pmatrix}\begin{pmatrix} \xi_0 \\ u_0 \\ b_0 \end{pmatrix} = \sigma\begin{pmatrix} \frac{1}{Pr} & 0 & 0 \\ 0 & \frac{1}{Pr} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \xi_0 \\ u_0 \\ b_0 \end{pmatrix}$$

giving the dispersion relation

$$(\frac{\sigma}{Pr} + A)^2(\sigma + A) + (\sigma + A)\frac{\pi^2}{A}T - (\frac{\sigma}{Pr} + A)\frac{k^2}{A}Ra = 0$$

discussed previously.

Fluxes

Note that linear theory predicts a buoyancy flux

$$wb = \frac{\sigma + \kappa A}{|S|}b_0^2\cos^2(ky)\sin^2(mz)e^{2\sigma t}$$

which has a non-zero and positive average

$$\langle wb\rangle = \frac{\sigma + \kappa A}{2|S|}b_0^2\sin^2(mz)e^{2\sigma t}$$

implying that the instability will draw energy from the potential energy field and convert it to kinetic energy (motion).

Lorenz eqns. and chaos

To derive the Lorenz equations, we note that mean buoyancy satisfies

$$\frac{\partial}{\partial t}\langle b\rangle = -\frac{\partial}{\partial z}\langle w'b'\rangle + \frac{\partial^2}{\partial z^2}\langle b\rangle$$

suggesting a correction to the mean profile proportional to $\sin(2\pi z)$. On the other hand, the mean vertical momentum flux is zero. So we try representing the fields as

$$b = -X(t)\sin(2\pi z) - Z(t)\cos(ky)\sin(\pi z)$$
$$\xi = Y(t)\sin(ky)\sin(\pi z)$$
$$u = U(t)\sin(ky)\cos(\pi z)$$

We substitute these into the equations and project by the various coefficients to get a dynamical system:

$$\frac{\partial}{\partial t}X = \frac{k\pi}{A}YZ - 4\pi^2 X$$
$$\frac{1}{Pr}\frac{\partial}{\partial t}Y = kRaZ - AY - \pi TU$$
$$\frac{\partial}{\partial t}Z = \frac{k}{A}Y - \frac{k\pi}{A}XY - AZ$$
$$\frac{1}{Pr}\frac{\partial}{\partial t}U = \frac{\pi}{A}Y - AU$$

If we rescale $\frac{\partial}{\partial t} \to A\frac{\partial}{\partial t}$ and choose suitable scales for the variables, we can get the classical Lorenz system (with an extra equation for rotation)

$$\frac{\partial}{\partial t}X = YZ - \beta X$$
$$\frac{1}{Pr}\frac{\partial}{\partial t}Y = Z - Y - U$$
$$\frac{\partial}{\partial t}Z = \rho Y - XY - Z$$
$$\frac{1}{Pr}\frac{\partial}{\partial t}U = \tau Y - U$$

with

$$\beta = \frac{4\pi^2}{A} \quad , \quad \rho = \frac{Ra}{A^3/k^2} \quad , \quad \tau = \frac{T}{A^3/\pi^2}$$

The growth rate is given by

$$(\frac{\sigma}{Pr} + 1)^2(\sigma + 1) + (\sigma + 1)\tau - (\frac{\sigma}{Pr} + 1)\rho = 0$$

which is a rescaled version of the previous result. The neutral curve is just

$$\rho = \tau + 1$$

Graphics: Lorenz eqns tau=0 rho=0.9 rho=2 rho=20 same rho=24.1 same rho=24.2 same closeup rho=25 same sensitivity closeup Graphics: Lorenz eqns tau=10 rho=10 rho=20 rho=40 same rho=80 same closeup medium rho rho=350

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Bifurcations

The steady solution $X = Y = Z = 0$ for $\tau = 0$ becomes unstable for $\rho > 1$, leading to a second steady solution $X = \rho - 1$, $Y = Z = \sqrt{\beta(\rho - 1)}$. This solution becomes unstable at $\rho = Pr(Pr + \beta + 3)/(Pr - \beta - 1) = 24.737$. This is a **subcritical** Hopf bifurcation. For larger ρ, the 3D system has neither stable 1D equilibria nor stable 2D limit cycles.

http://www.atm.ox.ac.uk/user/read/chaos/lect6.pdf

Attractor

If we consider the probability of being in a particular volume in (X, Y, Z, U) space, it evolves by

$$\frac{\partial}{\partial t}\mathcal{P} = -\frac{\partial \dot{X}\mathcal{P}}{\partial X} - \frac{\partial \dot{Y}\mathcal{P}}{\partial Y} - \frac{\partial \dot{Z}\mathcal{P}}{\partial Z} - \frac{\partial \dot{U}\mathcal{P}}{\partial U}$$

$$= -\frac{\partial Pr(YZ - \beta X)\mathcal{P}}{\partial X} - \frac{\partial (Z - Y - U)\mathcal{P}}{\partial Y} - \frac{\partial (\rho Y - XY - Z)\mathcal{P}}{\partial Z} - \frac{\partial Pr(\tau Y - U)\mathcal{P}}{\partial U}$$

Note that the "velocity" in phase space is convergent

$$\frac{\partial \dot{X}}{\partial X} + \frac{\partial \dot{Y}}{\partial Y} + \frac{\partial \dot{Z}}{\partial Z} + \frac{\partial \dot{U}}{\partial U} = -Pr\beta - 2 - Pr < 0$$

so that the volume continually contracts. Thus the solutions reside on an attractor with dimension less than 3; calculations suggest $D \sim 2.062$ (Sprott, J. 1997). He also estimates the Lyapunov exponents to be 0.906, -14.572.

http://sprott.physics.wisc.edu/chaos/lorenzle.htm

Mean Field Approx.

The mean field approximation (Herring, 1963, *J. Atmos. Sci.*) works with the equation for the mean buoyancy

$$\frac{\partial}{\partial t}\langle b \rangle = -\frac{\partial}{\partial z}\langle w'b' \rangle + \frac{\partial^2}{\partial z^2}\langle b \rangle$$

and the fluctuation flow

$$\frac{1}{Pr}\frac{\partial}{\partial t}\xi + \frac{1}{Pr}\nabla \cdot (\mathbf{u}\xi) = T\frac{\partial}{\partial z}u + Ra\frac{\partial}{\partial y}b' + \nabla^2\xi$$

$$\nabla^2\phi = \xi$$

$$\frac{\partial}{\partial t}b' + \nabla \cdot (\mathbf{u}'b' - \hat{\mathbf{z}}\langle w'b' \rangle) = -w'\frac{\partial}{\partial z}\langle b \rangle + \nabla^2 b'$$

The mean-field approx. involves dropping the nonlinear terms in the fluctuation equations, so that they revert to the linear stability problem — except that the mean $\langle b \rangle$ is changing with time and is more complicated than $-z$ in the vertical. The vertical structures of the perturbations will also change with time. However, the perturbation equations can still be separated with horizontal structures $\nabla^2 b' = -k^2 b'$, and we choose one or more k values.

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Note that there is no real requirement that the system be two-D; planforms like those shown previously are perfectly acceptable.

The measure of the effects of the fluctuating/ turbulent velocities is the **Nusselt number**, which is the ratio of the heat (or buoyancy) transport to that carried by conduction alone

$$Nu = \frac{\langle w'b' \rangle - \kappa \frac{\partial}{\partial z} \langle b \rangle}{-\kappa \Delta b/h}$$

In general, we need to average over long times as well.

    Graphics: mfa 2*crit 10*crit 50*crit 50*crit,1,2,3 Nu

Full Solutions 2D

    Graphics: 2d 2*crit Nu 10*crit Nu 50*crit Nu 50*crit wide 75*crit
Nu Nu 95 100*crit Nu 200*crit Nu bbar 500*crit Nu bbar b range w'b'
Nu-Ra

3D

Measurements at high Rayleigh number are difficult and fraught with problems from side walls, uneven temperature on the boundaries, non-Boussinesq effects, etc. But they tend to show $Nu \sim Ra^{0.29-0.3}$.

Takeshita, T., T.Segawa, J.A.Glazier, and M.Sano (1996) Thermal turbulence in mercury. *Phys. Rev. Let.*, **76**, 1465-68.

Ahlers, G. and X.Xu Prandlt-number dependence of heat transport in turbulent Rayleigh-Benard convection.

Nikolaenko, A. and G.Ahlers (2003) Nusselt number measurements in turbulent Rayleigh-Bènard convection. *Phys. Rev. Let.*, **91**

    Graphics: 3d variance in T Nu Nu Nu

Large Rayleigh number scalings

The 2D calculations and 3D experiments suggest $Nu \sim Ra^{\alpha}$. There are various arguments for what the power law should be:
1) The buoyancy (heat) flux should become independent of the values of ν and κ. In 3D turbulence, the flux of energy down the spectrum is given by the rate of injection; viscosity only determines the scale at which it is finally dissipated. If the same idea were to hold in convection then

$$Nu = \frac{F_b}{(\kappa \Delta b / h)} \to \frac{1}{\kappa} \sim [Ra \, Pr]^{1/2}$$

This value of $\alpha = 0.5$ is much higher than observed. But see Roche, Castaing, Chabaud, and Hébral (2001, *Phys. Rev. E*, **63**, 45303). `Graphics: Rough sur-faces Apparatus Nu Nu`
2) Suppose that the final state looks like a broken line profile and that the sharp temperature gradients at the boundaries are nearly neutrally stable. If the boundary layer has thickness h_b, then

$$\frac{\frac{1}{2} \frac{\Delta b}{h_b} h_b^4}{\nu \kappa} = Ra_c \quad , \quad \text{(a constant)}$$

Therefore

$$Ra \left(\frac{h_b}{h} \right)^3 = 2 Ra_c$$

so that the boundary layer thickness decreases as $Ra^{-1/3}$. If the flux through the boundary layer is conductive (marginally stable)

$$Nu = \frac{\kappa \frac{1}{2} \Delta b}{h_b} \bigg/ \frac{\kappa \Delta b}{h} = \frac{1}{2} \frac{h}{h_b} \sim Ra^{1/3}$$

This works up to $Ra = 4 \times 10^7$ (result from Libchaber; see Khurana, A. (1988) *Phys. Today*, **41**, 17-20). Herring found this with the mean-field approach as well.
3) From Kadanoff, et al. we assume that

$$|b'| \sim \Delta b \, Ra^{\beta} \quad , \quad |w'| \sim \frac{\kappa}{h} Ra^{\omega}$$

Since the heat is carried by convection in the interior, this suggests

$$Nu \simeq \frac{|w'||b'|}{\kappa \Delta b / h} \sim Ra^{\beta + \omega}$$

so that $\alpha = \beta + \omega$. In the interior, we assume that the balance is between vertical advection of w and buoyant forces

$$w \frac{\partial}{\partial z} w \sim b' \quad \Rightarrow \quad |w'|^2 \sim |b'| h$$

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so that

$$\frac{\kappa^2}{h^2} Ra^{2\omega} \sim \Delta b h Ra^\beta \quad \Rightarrow \quad Ra^{2\omega} \sim Ra^{1+\beta}$$

giving $2\omega = \beta + 1$. Finally, assume that the buoyant forces in the boundary layer balance dissipation

$$\Delta b \sim \nu \frac{|w'|}{h_b^2} \quad \Rightarrow \quad Ra \sim \left(\frac{h}{h_b}\right)^2 Ra^\omega \quad \Rightarrow \quad Nu = \frac{h}{h_b} \sim Ra^\alpha \sim Ra^{\frac{1}{2}-\frac{1}{2}\omega}$$

giving $\alpha = \frac{1}{2} - \frac{1}{2}\omega$. Combining these three gives

$$\alpha = \frac{2}{7} \quad , \quad \omega = \frac{3}{7} \quad , \quad \beta = -\frac{1}{7}$$

This agrees well with experiments in the higher Rayleigh number range.

For a good general reference, see Kadanoff, L. (*Physics Today*, Aug. 2001) and Emanuel, K. (1994) *Atmospheric Convection*.

Mixed layer

In the atmosphere and oceans, one common cause of convection is heat fluxes at the surface which warm (atmosphere) or cool (ocean) the fluid near the boundary. The rest of the fluid is stably-stratified. Conceptually, the unstable region diffuses into the stable region, with both the buoyancy jump and the thickness increasing (practically, of course, the boundary layer always has some level of turbulence, so the transport of heat is likely to be related to that rather than to molecular processes). In a very short time, the Rayleigh number exceeds the critical value and convection begins. We'll consider the oceanic case; just turn upside down (and think of b as proportional to the potential temperature) for the atmosphere. The diffusive layer will grow as $h \sim \sqrt{\kappa t}$. If the buoyancy flux out the surface is Q, the effective buoyancy difference across layer is order Qt/h (ignoring the deep stratification); therefore the Rayleigh number grows as t^2.

    Graphics: Finite amp convection into stratification means waterfall
shallow layer means waterfall rotating means waterfall equilibrated
means

When convection is vigorous, the unstably-stratified part of the water column mixes to become essentially uniform. For fixed temperature boundary conditions, you then develop thin layers near the boundary with thickness such that the local Rayleigh number is nearly critical. If the heat flux is fixed, these layers do not occur and the temperature gradient decreases to small values.

The first examples illustrate the development when we start with an unstable layer over a stable layer, with no heat flux at the surface. Then we can figure out the depth of convection by finding the depth h such that

$$\overline{T}(-h, 0) = \frac{1}{h} \int_{-h}^{0} \overline{T}(z, 0)$$

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For the profile used in the numerical experiments ($\frac{\partial \overline{T}}{\partial z}$ in the upper h_0/h of the domain given by $-2\times$ the value in the lower portion), we find that the mixing depth is $\sqrt{\frac{3}{2}}h_0$. This corresponds to $h = 0.6$ and 0.3 in the two cases; these estimates are consistent with the experiments. This assumes that the convective elements near the end of the experiment do not penetrate significantly below the depth

The effect of surface cooling will be to mix the fluid until the buoyancy gradient becomes zero again. We can find the new profile by figuring out the depth h such that the heat content change balances the surface cooling:

$$\rho_0 c_p \int_{-h}^{0} dz \left[T(z) - T(-h) \right] = Q \quad (>0)$$

If we begin with a linearly increasing temperature towards the surface $T = T_0 + \gamma z$, we find the amount of heat removed by the time the mixed layer reaches depth h is $\rho_0 c_p \gamma h^2 / 2$.

- If the cooling rate is constant, it takes a time Qt to remove the heat $\rho_0 c_p \gamma h^2 / 2$ above $z = -h$, implying that the depth of the mixed layer increases as $\sqrt{t}$.

Graphics: Surface convection mfa means 2D means final mfa,f=0.1 mfa,f=1 mfa,f=5 mfa,f=5 2D means final

Convective Plumes and Thermals

Thermals

Suppose we release a blob of buoyant fluid at the surface in the atmosphere or we have a continuous source such as a thermal vent or smokestack; what happens? Let's consider the blob case first.

If the blob were a buoyant object, it would accelerate upwards until it reaches its terminal velocity where drag matches the buoyancy force. If its buoyancy initially is b, its volume $\mathcal{V}$, and its velocity W, we could use Stokes law to describe the motion by

$$\frac{\partial}{\partial t} \mathcal{V}W = (b - \overline{b})\mathcal{V} - \frac{\nu}{h^2}\mathcal{V}W$$

or

$$\frac{\partial}{\partial t} W = (b - \overline{b}) - \frac{\nu}{h^2}W$$

where $\overline{b}$ is the buoyancy of the fluid outside the blob. In reality, the blob **entrains** fluid from outside: this alters its volume, its buoyancy, and its momentum. Let us denote the surface area of the blob by S and the rate at which fluid crossed that surface by w_e; then our volume, buoyancy, and momentum equations become

$$\frac{\partial}{\partial t} \mathcal{V} = w_e S$$

$$\frac{\partial}{\partial t} \mathcal{V}b = w_e S\overline{b}$$

$$\frac{\partial}{\partial t} \mathcal{V}W = (b - \overline{b})\mathcal{V}$$

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(assuming molecular exchanges are negligible) or

$$\frac{\partial}{\partial t}\mathcal{V} = w_e S$$

$$\frac{\partial}{\partial t}b = -\frac{w_e S}{\mathcal{V}}(b - \bar{b})$$

$$\frac{\partial}{\partial t}W = (b - \bar{b}) - \frac{w_e S}{\mathcal{V}}W$$

$$\frac{\partial}{\partial t}Z = W$$

with the last term giving the height of the thermal. In this sense, entrainment acts like viscous drag and conduction in that it damps the velocity or buoyancy back to the external values. However, we cannot expect the damping rate to be constant.

If we make the entrainment hypothesis that the turbulent velocities are proportional to relative velocity W and look at the case with a uniform exterior $\bar{b} = 0$

$$w_e = \alpha W$$

and apply a shape similarity assumption

$$S = \beta V^{2/3}$$

we find

$$V = [V_0^{1/3} + \alpha\beta Z]^3 \sim Z^3 \sim t^{3/2} \quad , \quad b = b_0 \frac{V_0}{V} \sim Z^{-3} \sim t^{-3/2}$$

$$W = b_0 \frac{V_0}{V}t \sim Z^{-1} \sim t^{-1/2} \quad , \quad Z \sim t^{1/2}$$

Graphics: Thermals Z vs t b vs t W vs. t Graphics: Thermals in stratification Z vs t b vs t W vs. t W vs. t linear

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Plumes

For this model, we idealize the cross-section as a circle and compute the steady state balances. We have

$$\frac{\partial}{\partial z}\pi r^2 W q = \alpha W 2\pi r q_e + \pi r^2 q_s$$

with q_e the external value and q_s the source/sink. Applying to mass ($q = q_e = 1$, $q_s = 0$), buoyancy ($q = b$, $q_e = N^2 z$, $q_s = 0$) , and momentum ($q = W$, $q_e = 0$, $q_s = b - N^2 z$) gives

$$\frac{\partial}{\partial z}(\pi r^2 W) = 2\pi \alpha W r$$

$$\frac{\partial}{\partial z}(\pi r^2 W b') = -\pi r^2 W N^2$$

$$\frac{\partial}{\partial z}(\pi r^2 W^2) = \pi r^2 b'$$

with $b' = b - N^2 z$ the local buoyancy anomaly.

In the case of an unstratified environment, we can again solve the equations; however, let's consider how the similarity solution works. We assume

$$r \sim z^r \quad , \quad W \sim z^b$$

and use the conservation of buoyancy flux to find

$$b' \sim z^{-w-2r}$$

From the mass and momentum equations, we find the exponents satisfy

$$2r + w - 1 - r + w \Rightarrow r = 1 \quad , \quad r \sim z^{-1}$$

$$2r + 2w - 1 = 2r - w - 2r \Rightarrow w = -\frac{1}{3} \quad , \quad W \sim z^{-1/3} \Rightarrow b' \sim z^{-5/3}$$

Graphics: Plumes r vs z b vs z W vs. z Graphics: Plumes in stratification r vs z b vs z W vs z

Inhomogeneous forcing

When the fluxes at the surface vary with latitude, the fluid must be in motion (there is a vorticity generation mechanism from $\nabla \times b\hat{\mathbf{z}}$). This can occur on small scales (convective chimneys) or on large scales (the Hadley cell or thermohaline circulation). Generally, you get weak convection on the cooling side, filling the domain with cold fluid which upwells into a advective-diffusive thermocline on the warming side.

To illustrate one of the connections between the large and small scales, consider the problem originally examined in the lab by Rossby (1965, *Deep-Sea Res.*). Let us represent the surface boundary conditions on b as exchange of heat with the atmosphere

$$\kappa \frac{\partial b}{\partial z} = \frac{1}{T}(b_a - b)$$

and solve the two-D problem with $b_a = b_0 \cos(\pi y/L)$. If we use lab-type scaling, we have again a Rayleigh number and a Prandtl number. If we want to use this rashly for the ocean, we'd add an aspect ratio $\lambda = H/L$ and perhaps a ratio of vertical to horizontal mixing coefficients.

We can solve the weak convection case fairly easily: for $\lambda = 1$, and taking $T \to 0$ for convenience, we have

$$b \simeq b_0 \cos(\pi y) \cosh(\pi z)/\cosh(\pi)$$

We can then solve for the vorticity

$$\nabla^2 \xi \simeq -Ra \frac{\partial}{\partial y} b = Ra\pi \sin(\pi y) \cosh(\pi z)/\cosh(\pi)$$

This will give a vorticity minimum in the upper water column, resulting in flow in the upper ocean towards the cold side, returning deep.

Graphics: small Ra vertical struct

This flow causes the descending cold water to narrow into a plume while the ascending water broadens. As the deep water fills up with cold fluid, the convection weakens.

Graphics: Single gyre b zeta psi variance   Graphics: Double gyre b zeta psi variance

Now consider the "anti-turbulence" theorem of Paparella and Young (*JFM*, 2002). Multiplying the buoyancy equation by z and averaging over the ocean and over a long time gives

$$\langle wb \rangle = \kappa \frac{1}{H} [\iint b(0) - b(-H)]$$

$$= \kappa \frac{1}{H} [\iint b_a - b(-H)]$$

$$< \kappa \frac{1}{H} [\iint b_a - min(b_a)]$$

- The vertical heat flux is limited by the diffusivity

The kinetic energy satisfies

$$0 = \langle wb \rangle + \frac{1}{H}\iint \mathbf{u}(0)\cdot\tau - \nu\langle|\frac{\partial u_i}{\partial x_j}|^2\rangle$$

so that the dissipation is

$$\epsilon = \frac{1}{H}\iint \kappa\,[b(0) - b(-H)] + \mathbf{u}(0)\cdot\tau$$

$$< \frac{1}{H}\iint \kappa\,[b_a - min(b_a)] + \mathbf{u}(0)\cdot\tau$$

- In the absence of wind forcing the dissipation is **not** insensitive to the small scale dissipation coefficients. Thus, thermally driven circulation is not turbulent in the Kolmogorov sense of energy cascading to the dissipation scales and maintaining the same dissipation rate even as the viscosity/ conductivity gets smaller.

With winds, the required dissipation is more or less independent of the viscosity and the flow will become turbulent.

The circulation strength is also limited: Siggers (2002) shows that a measure of the horizontal flux of heat is bounded by

$$Nu \sim \frac{|\langle vb\rangle|}{|\kappa\frac{\Delta b(0)}{L}|} < \left(\frac{\Delta b(0)L^3}{\nu\kappa}\right)^{1/3}$$

so that the advective heat fluxes and velocities also become small as ν and κ become small.

These versions of Sandstrom's theorem confirm Wunsch's view that the thermohaline circulation is driven not by the surface buoyancy gradients but by mixing in the deep ocean associated with winds and tides.

Other problems

Another important form convection takes in the atmosphere and ocean is gravity currents, where dense fluid is running down a slope. Using plume theory gives

$$\frac{\partial}{\partial x}hU = v_e$$

$$\frac{\partial}{\partial x}hU^2 = h(b - b_e)\sin\theta$$

$$\frac{\partial}{\partial x}hUb = v_e b_e$$

but the entrainment velocity now depends on Kelvin-Helmholtz instability of the top edge. Price and Baringer (1994, *Prog. Oceanogr.*, **33**, 161-200) use a form

$$v_e = \frac{0.08 - 0.1Ri}{1 + 5Ri}U \quad , \quad Ri = \frac{(b - b_e)h\cos\theta}{U^2} < 0.8$$