Vertical energy fluxes

From the linearized equations, we have the solutions

$$\begin{aligned}
w &= W\cos\theta \\
u &= -\frac{m}{k}W\cos\theta \\
v &= \frac{f}{\omega}\frac{m}{k}W\cos\theta \\
b &= \frac{N^2}{\omega}\sin\theta \\
p &= -\frac{1}{m}\frac{N^2-\omega^2}{\omega}W\cos\theta
\end{aligned}$$

with k the Horizontal wavenumber, $\theta = kx+mz-\omega t$, with $m(z)$ the slowly-varying vertical wavenumber, and

$$\omega^2 = \frac{k^2N^2+m^2f^2}{K^2} \quad , \quad K^2 = k^2+m^2$$

Direct

We calculate

$$\langle wp\rangle = -\frac{1}{2}\frac{N^2-\omega^2}{m\omega}W^2$$

where N^2 is varying; to simplify, we use

$$(N^2-\omega^2)k^2 = (\omega^2-f^2)m^2$$

and get

$$\langle wp\rangle = -\frac{1}{2}\frac{\omega^2-f^2}{\omega k^2}mW^2$$

Since ω, f, and k do not change along the ray, constant flux implies mW^2 is constant, or $W \sim m^{-1/2}$.

Group velocity

For the vertical group velocity, we have

$$2\omega c_g = \frac{\partial}{\partial m}\omega^2 = \frac{\partial}{\partial m}\frac{N^2k^2 + f^2m^2}{k^2+m^2} = 2m\frac{f^2}{K^2} - 2m\frac{\omega^2}{K^2}$$

so that

$$c_g = \frac{m}{K^2}\frac{f^2-\omega^2}{\omega} = -\frac{m}{K^2}\frac{\omega^2-f^2}{\omega}$$

The energy is

$$E = \frac{1}{4}\left(\frac{m^2}{k^2} + \frac{f^2}{\omega^2}\frac{m^2}{k^2} + 1 + \frac{1}{N^2}\frac{N^4}{\omega^2}\right)W^2 = \frac{1}{4}\left(\frac{K^2}{k^2} + \frac{f^2m^2+N^2k^2}{k^2\omega^2}\right)W^2 = \frac{1}{4}\left(\frac{K^2}{k^2}+\frac{K^2}{k^2}\right)W^2$$

$$\Rightarrow \quad E = \frac{1}{2}\frac{K^2}{k^2}W^2$$

(combining the $u^2/2$ and $w^2/2$ and the $v^2/2$ and $b^2/2N^2$ and then using the dispersion relation in the latter). Thus we find, as before,

$$c_gE = -\frac{1}{2}\frac{m}{K^2}\frac{\omega^2-f^2}{\omega}\frac{K^2}{k^2}W^2 = -\frac{1}{2}\frac{\omega^2-f^2}{\omega k^2}mW^2$$