# 12.802
# Small Scale Ocean Dynamics

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## Internal gravity waves

Pedlosky, Waves in the Ocean and Atmosphere, Chapter 7.

### Group velocity

The group velocity of internal waves can be easily computed using spherical coordinates $(\kappa, \theta, \phi)$ sketched in Fig. 7.4 of Pedlosky's book,

$$\boldsymbol{c}_g = \nabla\omega = N\left(\frac{\partial}{\partial\kappa}, \frac{1}{\kappa}\frac{\partial}{\partial\theta}, \frac{1}{\kappa\sin\theta}\frac{\partial}{\partial\phi}\right)\cos\theta = -\frac{N}{\kappa}\sin\theta\,\hat{\boldsymbol{\theta}}.$$

It is then easy to show that the group velocity is orthogonal to the wave vector,

$$\boldsymbol{c}_g \cdot \boldsymbol{k} = 0.$$

## Internal gravity waves, group velocity and reflection

Pedlosky, Waves in the Ocean and Atmosphere, Chapter 8.

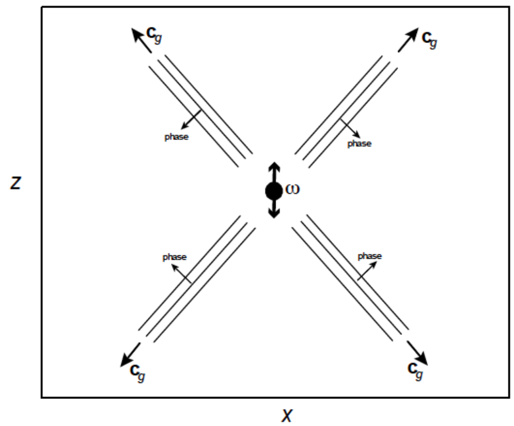



Figure 1: Radiation of waves from an oscillating source..

# Inertia-gravity waves

Pedlosky, Waves in the Ocean and Atmosphere, Chapter 11.

# WKB theory for inertia-gravity waves

Pedlosky, Waves in the Ocean and Atmosphere, Chapter 9.

Pedlosky's notes describe the non-rotating problem for internal gravity waves. The addition of rotation is trivial and affects only the definition of vertical wavenumber, which satisfies the dispersion relationship,

$$m^2(z) = \frac{N^2(z) - \omega^2}{\omega^2 - f^2} \kappa_h^2.$$

# Internal gravity wave spectrum

Pedlosky, Waves in the Ocean and Atmosphere, Chapter 9, pages 75-79.

Wunsch, Modern Observational Physical Oceanography, Chapter 5.

The GM spectrum is written as a sum of standing mode. Standing modes imply no vertical propagation of energy consistent with the model assumption that the upward and downward internal wave energy propagation are approximately equal. The spectrum is horizontally isotropic as it depends only on the horizontal wavenumber. The dependence of the spectrum on $N(z)$ is consistent with WKB scaling. In chapter 9 of Pedlosky's book, we have seen that $A(z)m^{1/2}$ is approximately constant for modes whose vertical scale is smaller than the scale over which $N(z)$ varies. (Remember that $A(z)$ is the amplitude of the vertical velocity in Pedlosky's notation. Hence for these modes we ought to expect that the spectrum of the vertical velocity and vertical displacement scale with $m^{-1}$. We know that in the WKB approximation $m \propto (N^2 - \omega^2)^{1/2}$ or $m \propto N(z)$ for waves with frequencies above $N$. The GM spectrum applies only to waves with $\omega \ll N$, because it was derived for hydrostatic waves.

The GM spectrum can be used to calculate the mean-square vertical displacements and kinetic energy which are in accord with typical observations,

$$\langle \zeta^2 \rangle = \int \mathrm{d}\omega \sum_n \Phi_{VD}(\omega, n) = \frac{1}{2} b^2 E_0 \frac{N_0}{N(z)} = 53 \frac{N_0}{N(z)} \mathrm{m}^2$$

and

$$\langle u^2 + v^2 \rangle = \int \mathrm{d}\omega \sum_n \Phi_{KE}(\omega, n) = \frac{3}{2} b^2 E_0 N_0 N(z) = 44 \times 10^{-4} \frac{N(z)}{N_0} \mathrm{m}^2\ \mathrm{s}^2.$$

# Topographic radiation of internal waves by mean flows

Pedlosky, Waves in the Ocean and Atmosphere, Chapter 10.

Topographic radiation of waves by tidal flows

• equations of motions are like before

$\tilde{D}^2 \nabla^2 \psi + N^2 \psi_{xx} = 0$ requires $\frac{u}{U} \ll 1$ (linearization)

$\psi = U h(x)$ @ $z = 0$ requires $\frac{h}{\text{wave scale}} \ll 1$ (small topographic amplitude)

N.B. $w = (\overset{\nearrow 0}{u} + U) h_x$ @ $z = h$

$\underset{\substack{\uparrow \\ \text{smaller}}}{w(0)} + \frac{\partial w}{\partial z} h = U h_x$

• new ingredient $U = U_0 \cos \omega t$

• as before $h = h_0 \cos Kx$

• solutions will have the form $\psi = \phi(z) \cos Kx \, e^{i\omega t}$ if

small excursion parameter : $\frac{U_0 K}{\omega} \ll 1$

$U_0 \simeq 4$ cm/sec (deep tidal velocity)
$\omega \simeq 2\pi / 12$ hours $\simeq 1.4 \times 10^{-4}$ s$^{-1}$
$U_0/\omega \simeq 280$ m $\ll$ scale of Mid-Atlantic ridge

$\Rightarrow \tilde{D} = \underset{[\omega]}{\partial_t} + \underset{[UK]}{U \partial_x} \simeq \partial_t$

- $-\omega^2\left(-K^2\phi+\phi_{zz}\right)-N^2K^2\phi=0$

$$\phi_{zz}+\left(\frac{N^2}{\omega^2}K^2-K^2\right)\phi=0$$

$\phi = Uh_0 \quad @\ z=0$

$\phi \to 0$ as $z\to\infty$ or radiation condition

- <u>case 1 - $N^2<\omega^2$</u>

$$\phi = Ae^{mz}+Be^{-mz}$$

$$m=\frac{\sqrt{\omega^2-N^2}}{\omega}K$$

$A=0$

$B=Uh_0$

$$\psi = Uh_0 e^{-mz}\cos Kx\cos\omega t = e^{-mz}h(x)\,U(t)$$

- case 2 - $N^2>\omega^2$

$$\phi = Ae^{imz}+Be^{-imz}$$

$$m=\frac{\sqrt{N^2-\omega^2}}{\omega}K$$

$\psi_A = U_0h_0\cos(\omega t+mz)\cos Kx$ | $\psi_B = Uh_0\cos(\omega t-mz)\cos Kx$

$\overline{pw} = +\overline{p\psi_x} = -\overline{p_x\psi}$

but

$u_t = -\frac{1}{\rho_{ref}}p_x \Rightarrow p_x = +\rho_{ref}\psi_{zt}$

$\overline{pw} = -\rho_{ref}\,\overline{\psi\psi_{zt}}$

N.B.

$$\overline{up} = -\overline{\psi_z p} = \overline{\psi p_z}$$

$$-\frac{1}{\rho_{ref}} p_z = \frac{\partial w}{\partial t} - b = \int dt \left( \frac{\partial^2 w}{\partial t^2} + N^2 w \right) = \frac{N^2 - \omega^2}{\omega} \sin(\omega t + mz) \cos Kx \times U_0 h_0$$

$$\overline{up} = \overline{\psi \psi_z} = \frac{N^2 - \omega^2}{\omega} \overline{\sin(\omega t + mz) \cos(\omega t + mz) \cos^2 Kx \times U_0^2 h^2}$$

$$= \frac{1}{2} \frac{N^2 - \omega^2}{\omega} \sin(2(\omega t + mz))$$

$$\overline{pw} = -\rho_{ref} U_0^2 h_0^2 (-\omega m) \frac{1}{4} = \frac{1}{4} \rho_{ref} U^2 h_0^2 \omega m \geq 0$$

$$\overline{pw} = -\rho_{ref} U_0^2 h_0^2 \omega m \frac{1}{4} \leq 0$$

- $\overline{pw} = \frac{1}{4} \rho_{ref} U^2 h_0^2 \sqrt{N^2 - \omega^2}\, K$

- compare with lee-wave solution

$$\overline{pw} = \frac{1}{2} \rho_{ref} U^2 h_0^2 \sqrt{N^2 - K^2 U^2}\, K$$

- vertical scale of waves

$$m = \frac{\sqrt{N^2 - \omega^2}}{\omega} k \simeq \frac{N}{\omega} K \Rightarrow \lambda_v = \frac{\omega}{N} \lambda_H = \frac{10^{-4}}{10^{-3}} \lambda_H = 10^{-1} \lambda_H$$

horizontal scales $\lambda_H \geq 10 \lambda_v = 40\text{Km}$ feel top boundary

- low modes have vertical wavenumbers quantized $m H = n\pi$ (to satisfy $w = 0$ @ top and $w = Uh_x$ @ bottom)

- only horizontal wavenumbers $K = \frac{\omega}{\sqrt{N^2 - \omega^2}} \frac{n\pi}{H}$ will radiate, other modes will be forced barotropic or bottom-trapped baroclinic disturbances

- finite slopes do not change net radiation much (Balmforth et al. 2002)

=> Adding rotation would give

$$F = \begin{cases} \text{tides} & \frac{1}{4}\sqrt{\frac{\omega^2-f^2}{\omega^2}}\sqrt{N^2-\omega^2}\, U_0^2 K h_0^2 \\ \text{lee waves} & \frac{1}{2}\sqrt{K^2U_0^2-f^2}\sqrt{N^2-K^2U_0^2}\, U_0 h_0^2 \end{cases}$$

· tides radiate as long as $f < \omega < N$

$$\underbrace{N > \frac{2\pi}{N\text{hours}}}_{\text{almost always satisfied}} > 2 \cdot \frac{2\pi}{24\text{h}} \sin\theta$$

$$\sin\theta \leq \frac{12\,\text{hours}}{N\text{hours}}$$

$$M2: \quad \sin\theta < \frac{12}{12\frac{25}{60}} \Rightarrow \theta < 74.5 \text{ degrees}$$

$$K1: \quad \sin\theta < \frac{12}{23.93} \Rightarrow \theta < 30 \text{ degrees}$$

Generation of near-inertial waves in the ML

- linearized ML dynamics

$$\begin{cases} \dfrac{\partial u}{\partial t} - fv = -\dfrac{1}{\rho_{ref}}\dfrac{\partial p}{\partial x} + \dfrac{1}{\rho_{ref}}\dfrac{\partial \tau^{(x)}}{\partial z} \\ \\ \dfrac{\partial v}{\partial t} + fu = -\dfrac{1}{\rho_{ref}}\dfrac{\partial p}{\partial y} + \dfrac{1}{\rho_{ref}}\dfrac{\partial \tau^{(y)}}{\partial z} \end{cases}$$

- in a bulk model of the ML we assume $\underset{\sim}{u}_h = \underset{\sim}{u}_h(x,y,t)$ and separate $\underset{\sim}{u}_h = \underset{\sim}{u}_E + \underset{\sim}{u}_P$

$$\frac{\partial \underset{\sim}{u}_P}{\partial t} + f\hat{z}\times\underset{\sim}{u}_P = -\frac{1}{\rho_{ref}}\nabla_h p$$

$$\frac{\partial \underset{\sim}{u}_E}{\partial t} + f\hat{z}\times\underset{\sim}{u}_E = \frac{1}{\rho_{ref}}\frac{\partial \underset{\sim}{\tau}}{\partial z}$$

- focusing on wind response we solve for $\underset{\sim}{u}_E$

$$h\left[\frac{\partial \underset{\sim}{u}_E}{\partial t} + f\hat{z}\times\underset{\sim}{u}_E\right] = \frac{\underset{\sim}{\tau}^W - \underset{\sim}{\tau}^B}{\rho_{ref}}$$

- to represent damping by wave-wave interactions and friction

$$\frac{\partial \underset{\sim}{u}_E}{\partial t} + f\hat{z}\times\underset{\sim}{u}_E = \underset{\sim}{T} \qquad , \underset{\sim}{T} = \frac{(\tau^{(x)}, \tau^{(y)})}{\rho_{ref}\, h}$$

- Case 1: response to a steady wind

$$\underset{\sim}{u}_E = -\frac{1}{f}\hat{z}\times\underset{\sim}{T} \qquad \text{(Ekman solution)}$$

• Case 2: response to a rotating wind

$$\begin{cases} T^{(x)} = T_0 \cos\omega t \\ T^{(y)} = T_0 \sin\omega t \end{cases}$$

if $\omega > 0$ wind rotates cyclonically (CCW)

if $\omega < 0$ " " anticyclonically (CW)

• write solutions in the form

$Z = u + iv$

$T = T^{(x)} + i\,T^{(y)}$

add damping

$$\frac{dZ}{dt} + ifZ = T - rZ = \hat{T}\, e^{i\omega t} - rZ$$

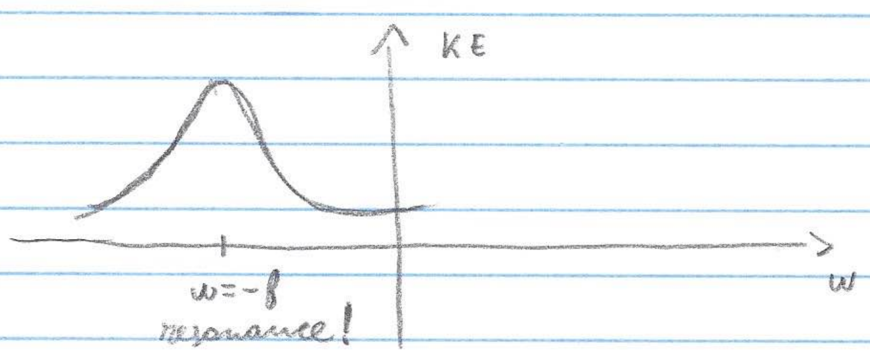
solutions have the form $Z = Z_0 e^{i\omega t}$

$(i\omega + if - r)\,\hat{Z} = \hat{T}$

$$\hat{Z} = \frac{\hat{T}}{i(\omega+f) - r}\, e^{i\omega t}$$

• the kinetic energy generated by the swirling wind is

$$KE = \frac{1}{2}|Z|^2 = \frac{|\hat{T}|^2}{r^2 + (\omega+f)^2}$$




• $$\underset{\sim}{c}_g = \nabla\omega = \frac{1}{k}\frac{\partial}{\partial\theta}\sqrt{N^2\cos^2\theta + f^2\sin^2\theta} = \frac{1}{2k\omega}\left(-2N^2\cos\theta\sin\theta + 2f^2\sin\theta\cos\theta\right)\hat{\theta} =$$

$$= -\frac{1}{2k\omega}(N^2 - f^2)\sin 2\theta \quad (=0 \text{ for } \theta = \tfrac{\pi}{2})$$

- NIO radiate because local $f$ is modulated by $\beta$, eddies, ...

- general perturbation

$$\frac{\partial \underset{\sim}{u}_E}{\partial t} + f\hat{z}\times\underset{\sim}{u}_E = \underset{\sim}{T}$$

for $\underset{\sim}{T} = \int d\underset{\sim}{K} \int d\omega \; \hat{\underset{\sim}{T}}(\omega, K)\, e^{i(\underset{\sim}{K}_h\cdot\underset{\sim}{x} - \omega t)}$

$$\hat{u}_E = \frac{i\omega\,\hat{T}^{(x)} - f\,\hat{T}^{(y)}}{\omega^2 - f^2}$$

$$\hat{v}_E = \frac{i\omega\,\hat{T}^{(y)} + f\,\hat{T}^{(x)}}{\omega^2 - f^2}$$

$$\hat{w}_E = -\underset{\sim}{K}_h\cdot\hat{\underset{\sim}{u}}_E = -\frac{\omega\,\underset{\sim}{K}_h\cdot\hat{\underset{\sim}{T}} + if\,\underset{\sim}{K}_h\times\hat{\underset{\sim}{T}}}{\omega^2 - f^2}$$

- imposing the b.c. on $w$ at top of ocean interior, I solve for waves radiating into stratified ocean

# Inertia-gravity waves and balanced motions

Pedlosky, Waves in the Ocean and Atmosphere, Chapter 11.