12.802
Small Scale Ocean Dynamics

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Basic fluid equations

The momentum, mass, heat and salinity budgets for an ocean parcel on Earth are,

$$\rho \frac{D\boldsymbol{u}}{Dt} + f\rho\hat{\boldsymbol{z}} \times \boldsymbol{u} = -\nabla p - g\rho\hat{\mathbf{z}} + \mathcal{F}, \tag{1}$$

$$\frac{D\rho}{Dt} = -\rho\nabla \cdot \boldsymbol{u}, \tag{2}$$

$$\rho \frac{D\theta}{Dt} = \mathcal{G}_\theta, \tag{3}$$

$$\rho \frac{DS}{Dt} = \mathcal{G}_S, \tag{4}$$

where D/Dt is the substantial time derivative, $\boldsymbol{u}$ is the 3D velocity field, f is the Coriolis frequency, g is the gravity acceleration, ρ is density, p is pressure, θ is potential temperature and S is salinity. Forcing of momentum, temperature and salinity are $\mathcal{F}$, $\mathcal{G}_\theta$ and $\mathcal{G}_S$.

This is a set of six equations for seven variables $(\boldsymbol{u}, p, \rho, \theta, S)$. The system is completed by specification of an equation of state $\rho \equiv \rho(\theta, S, p)$. Taking the substantial time derivative of the equation of state, we can rewrite the mass conservation equation as,

$$\frac{D\rho}{Dt} = \frac{\partial\rho}{\partial\theta}\frac{D\theta}{Dt} + \frac{\partial\rho}{\partial S}\frac{DS}{Dt} + \frac{\partial\rho}{\partial p}\frac{Dp}{Dt} = -\rho\nabla \cdot \boldsymbol{u}. \tag{5}$$

Introducing the expansion coefficient for heat $\alpha = -\rho^{-1}\partial\rho/\partial\theta$, the contraction coefficient for salt $\beta = \rho^{-1}\partial\rho/\partial S$, and the sound speed $c_s^2 = \partial p/\partial\rho$, we have,

$$\frac{1}{c_s^2}\frac{Dp}{Dt} = \alpha\mathcal{G}_\theta - \beta\mathcal{G}_S - \rho\nabla\cdot\boldsymbol{u}. \tag{6}$$

Linear waves

A simple view of the time-scale and types of motions exhibited by the set of six fluid dynamic equations for the ocean can be outlined by a linear wave analysis. Here we will focus on waves in the fluid interior and ignore, initially, the influence of the surface boundary conditions which gives rise to surface gravity waves. We follow closely the derivation in chapters 2 and 3 of Ocean Dynamics by Olbers, Willebrand and Eden, Ocean Dynamics (2012).

We will consider small adiabatic perturbations from a state of rest, the latter being characterized by a vertical stratification of salinity and potential temperature. Adiabatic in this context means no forcing in the salinity and potential temperature equations, i.e. $\mathcal{G}_\theta = \mathcal{G}_S = 0$. In order to have a basic state of rest, we must also drop any forcing in the momentum equation, $\mathcal{F} = 0$. All prognostic variables are expanded according to $a = a_r + a'$ (where a can be any one of u, v, w, p, S and θ) around a basic state defined as,

$$u_r = v_r = w_r = 0, \qquad \theta_r = \theta_r(z), \qquad S_r = S_r(z), \tag{7}$$

$$\frac{\partial p_r}{\partial z} = -g\rho_r, \qquad N_\theta^2 = g\alpha\frac{\partial\theta_r}{\partial z}, \qquad N_S^2 = -g\beta\frac{\partial S_r}{\partial z}. \tag{8}$$

Neglecting all terms which are quadratic or higher order in the perturbation quantities, one obtains,

$$\frac{\partial u'}{\partial t} - fv' + \frac{1}{\rho_r}\frac{\partial p'}{\partial x} = 0, \tag{9}$$

$$\frac{\partial v'}{\partial t} + fu' + \frac{1}{\rho_r}\frac{\partial p'}{\partial y} = 0, \tag{10}$$

$$\frac{\partial w'}{\partial t} + \frac{1}{\rho_r}\left(\frac{\partial p'}{\partial z} + \frac{g}{c_s^2}p'\right) + g(\beta S' - \alpha\theta') = 0, \tag{11}$$

$$\frac{1}{c_s^2}\frac{\partial p'}{\partial t} - \frac{g}{c_s^2}\rho_r w + \rho_r\nabla\cdot\boldsymbol{u'} = 0, \tag{12}$$

$$\frac{\partial\theta'}{\partial t} + \frac{N_\theta^2}{\alpha g}w' = 0, \tag{13}$$

$$\frac{\partial S'}{\partial t} - \frac{N_S^2}{\beta g}w' = 0. \tag{14}$$

The last two equations can be combined into one equation for potential buoyancy $b' = g(\alpha\theta' - \beta S')$ introducing the buoyancy frequency $N^2 = N_\theta^2 + N_s^2$. The speed of sound c_a is understood to be the speed of sound associated with the reference state $c_s = c_s(\theta_r, S_r, p_r)$. We can ignore terms proportional to c_s^2/g. This lengthscale is approximately 200 km for the sound speed in the ocean of $c_s \simeq 1500\text{m s}^{-1}$, much deeper than the full ocean depth $H = 4$ km. The terms proportional to c_s^2/g are smaller than leading order terms in the respective equations by a ratio Hg/c_s^2 (the ratio of the long surface gravity wave speed to the sound speed as we will see below). This approximation holds in the ocean, but not in the atmosphere where $H \sim 10$ km and $c_s \sim 300\text{ m s}^{-1}$. In terms of the perturbation equations, a small Hg/c_s^2 allows us to ignore the term $g/c_s^2 p'$ compared to $\partial p'/\partial z$, the term $g/c_s^2 w'$ compared to $\partial w'/\partial z$, and to assume that ρ_r is constant. With these simplifications the system of perturbation equations reduces to,

$$\frac{\partial u'}{\partial t} - fv' + \frac{1}{\rho_r}\frac{\partial p'}{\partial x} = 0, \tag{15}$$

$$\frac{\partial v'}{\partial t} + fu' + \frac{1}{\rho_r}\frac{\partial p'}{\partial y} = 0, \tag{16}$$

$$\frac{\partial w'}{\partial t} + \frac{1}{\rho_r}\frac{\partial p'}{\partial z} - b' = 0, \tag{17}$$

$$\frac{1}{\rho_r c_s^2}\frac{\partial p'}{\partial t} + \nabla \cdot \boldsymbol{u'} = 0, \tag{18}$$

$$\frac{\partial b}{\partial t} + N^2 w = 0. \tag{19}$$

It is convenient to introduce the horizontal streamfunction ψ and the velocity potential ϕ to replace the horizontal velocities u' and v',

$$u' \equiv \frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial y}, \qquad v' \equiv \frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial x}, \tag{20}$$

so that $\partial v'/\partial x - \partial u'/\partial y = \nabla_h^2 \psi$ and $\partial u'/\partial x + \partial v'/\partial y = \nabla_h^2 \phi$.

From Eqs. (15) through (19), one can derive the following equations (dropping all primes from now on)

$$\nabla_h^2 \frac{\partial \psi}{\partial t} + f\nabla_h^2 \phi = 0, \tag{21}$$

$$\nabla_h^2 \frac{\partial \phi}{\partial t} - f\nabla_h^2 \psi + \frac{1}{\rho_r}\nabla_h^2 p = 0, \tag{22}$$

$$\frac{\partial w}{\partial t} + \frac{1}{\rho_r}\frac{\partial p}{\partial z} - b = 0, \tag{23}$$

$$\frac{1}{c_s^2 \rho_r}\frac{\partial p}{\partial t} + \nabla_h^2 \phi + \frac{\partial w}{\partial z} = 0. \tag{24}$$

$$\frac{\partial b}{\partial t} + N^2 w = 0. \tag{25}$$

The first two equations are derived by taking the horizontal curl and divergence of (15) and (16). A number of approximations have been made to derive equations (21) through (25). We ignored lateral variations of the Coriolis parameter f, thereby filtering out Rossby waves. We also ignored subtleties related to the nonlinearities in the equation of state. For maximum simplicity, we will further assume that the stratification N^2 is constant. This last approximation is neither appropriate for typical oceanic conditions, nor necessary to make analytical progress, but it simplifies the algebra and allows us to focus on the key qualitative properties of oceanic waves. A more general derivation of the equations is posted on the class website.

As for any linear system with constant coefficients, linear wave solutions can be found in the form,

$$(\psi, \phi, w, p/\rho_r, b)^T = \boldsymbol{V}_0 e^{i(kx+ly+mz-\omega t)}, \tag{26}$$

where $\boldsymbol{V}_0$ is a constant vector and the waves have wavenumber $\boldsymbol{k} = (k, l, m)$ and frequency ω. Substituting this expression in the system (21) through (25) leads to a standard eigenvalue problem of the form,

$$A\boldsymbol{V}_0 = i\omega \boldsymbol{V}_0, \tag{27}$$

with the matrix,

$$\boldsymbol{A} = \left\{ \begin{array}{ccccc} 0 & f & 0 & 0 & 0 \\ -f & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & im & -1 \\ 0 & -c_s^2\kappa_h^2 & ic_s^2 m & 0 & 0 \\ 0 & 0 & N^2 & 0 & 0 \end{array} \right\}. \tag{28}$$

Here the notation $\kappa_h^2 \equiv k^2 + l^2$ and $\kappa^2 = k^2 + l^2 + m^2$ has been used. The eigenfrequencies ω are given by the roots of the characteristic equation,

$$\omega^5 - \omega^3(f^2 + N^2 + c_s^2\kappa^2) + \omega(c_s^2\kappa_h^2 N^2 + c_s^2 m^2 f^2 + N^2 f^2) = 0 \tag{29}$$

which has five roots ω_i for $i = 1 \dots 5$. One of these roots corresponds to zero frequency, say $\omega_5 = 0$, reflecting non oscillatory solutions in geostrophic ($\psi = p/\rho_r f$) and hydrostatic balance. Had we allowed for lateral variations in the Coriolis frequency, this root would have given slowly oscillating Rossby waves. For f constant, the frequency of Rossby waves, which is proportional to df/dy, becomes zero and the period of the oscillations infinite.

Three inverse timescales remain in the 5th order polynomial, which under typical oceanic conditions have *very* different magnitudes,

- the Coriolis frequency with a magnitude $f \sim 10^{-4}$ s^{-1} for midlatitudes,

- the buoyancy frequency $N \sim 10^{-3}$ s^{-1} for a typical stratification in the ocean thermoclines,
- the frequency of sound waves $c_s\kappa$ with magnitude 1.5×10^{-2}-1.5×10^{3} s^{-1} for wavelengths between $\kappa^{-1} \sim 1$ m to $\kappa^{-1} \sim 100$ km.

Under normal oceanic conditions we therefore have,

$$f \leq N \ll c_s\kappa,$$

where the $\ll$ sign is understood to denote a difference in magnitude by a factor,

$$\delta \sim \frac{N}{c_s\kappa} \leq 10^{-1},$$

depending on the scale of motion. It is straightforward to show that if $\delta \ll 1$, the polynomial (30) can be approximated by,

$$\omega^5 - \omega^3 c_s^2 \kappa^2 + \omega(c_s^2 \kappa_h^2 N^2 + c_s^2 m^2 f^2) = 0. \tag{30}$$

Because the relevant timescales are so different, approximate values for the roots of the polynomial (30) can be found quite easily. The bigger roots are approximately,

$$\omega_{1,2}^2 = c_s^2\kappa^2 + O(\delta^2),$$

and represent non dispersive acoustic waves. The smaller pair is instead given by,

$$\omega_{3,4}^2 = \frac{\kappa_h^2 N^2 + m^2 f^2}{\kappa^2} + O(\delta^2),$$

and constitutes the dispersion relationship of internal gravity waves. Finally the last and smallest root is simply

$$\omega_5 = 0 + O(\delta^2),$$

and represents Rossby waves in the limit $\beta \to 0$. Both for sound and internal gravity waves there exist two solutions each, because along any coordinate waves can propagate into both positive and negative directions. Rossby waves, on the other hand, have a phase propagation to the west only and hence only one root.

Influence of boundaries

Waves in a finite depth ocean must satisfy boundary conditions at the ocean bottom and surface. For a flat bottom ocean, the bottom boundary condition is,

$$w = 0, \qquad \text{at } z = -H. \tag{31}$$

The surface boundary condition is instead more complex,

$$w = \frac{\partial \eta}{\partial t} + \boldsymbol{u} \cdot \nabla \eta, \qquad p + p_r = p_{atm}, \qquad \text{at } z = \eta. \tag{32}$$

Linearizing the two surface boundary conditions simplifies to,

$$w = \frac{\partial \eta}{\partial t}, \qquad p(0) + p_r(0) - \rho_r g \eta = p_{atm}, \qquad \text{at } z = 0. \tag{33}$$

The two conditions can be combined into one by eliminating η,

$$w = \frac{1}{g\rho_r}\frac{\partial p}{\partial t}, \qquad \text{at } z = 0. \tag{34}$$

Solutions to the linearized equations (21) through (25) with the boundary conditions (31) and (48) can be written in the form

$$(\psi, \phi, w, p/\rho_r, b) = (\Psi(z), \Phi(z), W(z), P(z), B(z))e^{i(kx+ly-\omega t)}. \tag{35}$$

Substituting these solutions, the linearized equations can be reduced to an equation for $W(z)$,

$$\frac{d^2W}{dz^2} + \left(\kappa_h^2 \frac{N^2 - \omega^2}{\omega^2 - f^2} - \frac{N^2 - \omega^2}{c_s^2}\right) W = 0, \tag{36}$$

with boundary conditions,

$$W = 0, \qquad \text{at } z = -H, \tag{37}$$

$$\frac{dW}{dz} = \left(\frac{g\kappa_h^2}{\omega^2 - f^2} - \frac{g}{c_s^2}\right) W, \qquad \text{at } z = 0. \tag{38}$$

In the right hand side of the surface boundary condition, we will drop the term $-g/c_s^2 W$, because gH/c_s^2 is much smaller than W_z for oceanographic parameters.

Sound waves

Let us first consider waves with high frequencies, i.e. $\omega^2 \gg N^2 \geq f^2$. In this limit the equation (63) for $W(z)$ simplifies to,

$$\frac{d^2W}{dz^2} + \left(\frac{\omega^2}{c_s^2} - \kappa_h^2\right) W \simeq 0, \tag{39}$$

with boundary conditions,

$$W = 0, \qquad \text{at } z = -H, \tag{40}$$

$$\frac{dW}{dz} \simeq \frac{g\kappa_h^2}{\omega^2} W, \qquad \text{at } z = 0. \tag{41}$$

Interior modes

Solutions to equation (39) depend on the magnitude of ω^2. For $\omega^2 \geq c_s^2\kappa_h^2$, solutions are in the form of sines and cosines. Furthermore in this limit $g\kappa_h^2/\omega^2 \leq g/c_s^2$ and thus the boundary conditions reduce to,

$$W = 0, \qquad \text{at } z = -H, \tag{42}$$

$$\frac{dW}{dz} \simeq 0, \qquad \text{at } z = 0. \tag{43}$$

Given that $W(z)$ must vanish at $z = -H$, the solution can therefore be written as,

$$W(z) \simeq W_0 \sin\left(m(z + H)\right), \qquad \text{with } \omega^2 \simeq (\kappa_h^2 + m^2)c_s^2.$$

These are sound waves with the same frequency as the free waves discussed in the previous section. But the presence of a surface boundary condition selects a discrete set of possible wavenumbers m such that $W_z = 0$ at $z = 0$,

$$m \simeq \left(n + \frac{1}{2}\right)\frac{\pi}{H}.$$

The vertical wavenumbers are a discrete set that fits in the finite depth H.

We have shown that the smallness of gH/c_s^2 implies that to a good approximation the boundary condition reduces to imposing $W_z = 0$ at the ocean surface. Using the relationship between p and w based on the equations of motion, one can verify that this boundary condition is equivalent to setting p equal to a constant at the supper surface. Furthermore the smallness of the ratio gH/c_s^2 implies that the acoustic waves have higher frequency and phase speeds than long surface gravity waves. This helps to separate the two types of waves as we see next.

Surface modes

Let us now consider the case when $\omega^2 \leq c_s^2\kappa_h^2$. Solutions are in the form of hyperbolic sines and cosines. Given that $W(z)$ must vanish at $z = -H$, the solution can therefore be written as,

$$W(z) \simeq W_0 \sinh\left(m(z + H)\right), \qquad \text{with } \omega^2 \simeq (\kappa_h^2 - m^2)c_s^2.$$

The set of vertical wavenumbers are set by the surface boundary condition,

$$mH\cosh(mH) \simeq \frac{gH}{c_s^2}\frac{\kappa_h^2}{\kappa_h^2 - m^2}\sinh(mH). \tag{44}$$

Rearranging terms we find that,

$$\frac{\kappa_h^2 - m^2}{\kappa_h^2} \simeq \frac{gH}{c_s^2}\frac{\tanh(mH)}{mH}. \tag{45}$$

The function $\tan x/x$ is always smaller than one, and the coefficient $gH/c_s^2 \ll 1$. Thus the right hand side of the relationship in (46) is very small and at leading order $\kappa_h^2 \simeq m^2$. Substituting iteratively $\kappa_h^2 = m^2$ in the relationship, we find the second order correction,

$$\frac{\kappa_h^2 - m^2}{\kappa_h^2} \simeq \frac{gH}{c_s^2}\frac{\tanh(\kappa_h H)}{\kappa_h H}. \tag{46}$$

Substituting back in the dispersion relationship, we have,

$$\omega^2 \simeq g\kappa_h \tanh \kappa_h H,$$

the dispersion relationship of surface gravity waves.

Gravity waves

Let us now consider waves with frequencies ω^2 of order N. The vertical mode equation (63) in this limit reduces to,

$$\frac{d^2W}{dz^2} + \kappa_h^2 \frac{N^2 - \omega^2}{\omega^2 - f^2} W \simeq 0, \tag{47}$$

with boundary conditions approximated by,

$$W = 0, \qquad \text{at } z = 0 - H, \tag{48}$$

$$\frac{dW}{dz} = \frac{g\kappa_h^2}{\omega^2 - f^2} W, \qquad \text{at } z = 0. \tag{49}$$

Interior modes

For $(N^2 - \omega^2)/(\omega^2 - f^2) > 0$ solutions hare in the form of sines and cosines. These are oscillatory solutions and represent internal gravity waves. Imposing the boundary condition of $W = 0$ at $z = -H$ we find,

$$W(z) \simeq W_0 \sin\left(m(z + H)\right), \qquad \text{with } m^2 \simeq \kappa^2 \frac{N^2 - \omega^2}{\omega^2 - f^2}.$$

The top boundary condition for this solution reduces to,

$$\frac{m^2 f^2 + \kappa_h^2 N^2}{\kappa_h^2 + m^2} - f^2 \simeq \frac{g\kappa_h^2}{m} \tan mH \quad \Longrightarrow \quad \frac{H(N^2 - f^2)}{g} = H^2(\kappa_h^2 + m^2)\frac{\tan mH}{mH}, \tag{50}$$

which can be used to determine the vertical wavenumber m for any κ_h. The left hand side of (50) is the ratio of the ocean depth to the deformation radius which is a small number,

$$\epsilon \equiv \frac{H(N^2 - f^2)}{g} \simeq \frac{HN^2}{g} = \frac{H}{g}\frac{g}{\rho_0}\frac{\partial \rho}{\partial z} \simeq \frac{\Delta\rho}{\rho_0} \approx 10^{-3} \ll 1,$$

where $\Delta\rho$ is the density variation over the depth H. Perhaps it is more useful to think about HN^2/g as the ratio of the (square) of the maximum internal gravity wave frequency N^2 to the surface wave frequency (squared) for a wave whose wavelength is of the order of the depth of the fluid, g/H. The fact that this ratio is small thus implies that the surface waves have higher frequency and phase speeds than the internal waves. This helps to separate the two types of waves as we will see in the next subsection. For present purposes, the fact that HN^2/g is small implies that solutions to the boundary condition (50) are found at the zeros of the $\tan mH$,

$$mH = n\pi + O(\epsilon), \qquad n = 1, 2, \dots \qquad \text{(but not } n = 0\text{!)}$$

Up to terms of order ϵ, this corresponds to the solutions of (47) for rigid lid boundary conditions at both the top and bottom surfaces. Therefore, the error caused by imposing a *rigid rid approximation* is small, approximately 0.2%, for internal gravity waves.

Surface modes

For $(N^2 - \omega^2)/(\omega^2 - f^2) > 0$ solutions hare in the form of hyperbolic sines and cosines. Taking into account that the vertical velocity must vanish at $z = -H$, we can write the solution as,

$$W(z) \simeq W_0 \sinh\left(m(z + H)\right)$$

Substituting this solution in the surface boundary condition (50), one obtains,

$$W_0 \kappa_h \cosh \kappa_h H \simeq \frac{g\kappa_h^2}{\omega^2 - f^2} W_0 \sinh \kappa_h H$$

Substituting the dispersion relationship,

$$\omega^2 \simeq \frac{N^2\kappa_h^2 - f^2 m^2}{\kappa_h^2 - m^2},$$

we find that m must satisfy,

$$H^2(\kappa_h^2 - m^2)\frac{\tanh mH}{mH} \simeq \frac{H(N^2 - f^2)}{g} = \epsilon \ll 1.$$

Solutions to this equation exist for $|m| \simeq \kappa_h$. Solving iteratively, we thus have,

$$\kappa_h^2 - m^2 \simeq \frac{N^2 - f^2}{g}\frac{\kappa_h}{\tanh(H\kappa_h)}.$$

Substituting this expression in the dispersion relationship we find,

$$\omega^2 = f^2 + g\kappa_h \tanh H\kappa_h + O(\epsilon),$$

the dispersion relationship for rotating surface gravity waves.

Further reading: Pedlosky, Waves in the Ocean and Atmosphere, Chapter 10.

Appendix: Derivation of equation for $W(z)$

Assuming that solutions to the linearized equations (21) through (25) have the form

$$(\psi, \phi, w, p/\rho_r, b) = (\Psi(z), \Phi(z), W(z), P(z), B(z))e^{i(kx+ly-\omega t)}, \tag{51}$$

we obtain:

$$-i\omega\Psi + f\Phi = 0, \tag{52}$$
$$-i\omega\Phi - f\Psi + P = 0, \tag{53}$$
$$-i\omega W + \frac{\partial P}{\partial z} - B = 0, \tag{54}$$
$$-i\frac{\omega}{\rho_r c_s^2}P - \frac{g}{c_s^2}W - \kappa_h^2\Phi + \frac{\partial W}{\partial z} = 0, \tag{55}$$
$$-i\omega B + N^2 W = 0. \tag{56}$$

Multiplying the first equation by $-f$ and the second by $i\omega$ and summing them, we eliminate Ψ,

$$\Phi = -i\frac{\omega}{\omega^2 - f^2}P, \tag{57}$$
$$-i\omega W + \frac{\partial P}{\partial z} - B = 0, \tag{58}$$
$$-i\frac{\omega}{c_s^2}P - \kappa_h^2\Phi + \frac{\partial W}{\partial z} = 0, \tag{59}$$
$$-i\omega B + N^2 W = 0. \tag{60}$$

Multiplying the third equation by $i\omega$ and subtracting the fourth equation from it, we eliminate B,

$$(N^2 - \omega^2)W - i\omega\frac{\partial P}{\partial z} = 0, \tag{61}$$
$$-i\frac{\omega}{c_s^2}P + i\omega\frac{\kappa_h^2}{\omega^2 - f^2}P + \frac{\partial W}{\partial z} = 0, \tag{62}$$

where we also eliminated Φ from the last equation. Solving for W we obtain,

$$\frac{d^2W}{dz^2} + \left(\kappa_h^2\frac{N^2 - \omega^2}{\omega^2 - f^2} - \frac{N^2 - \omega^2}{c_s^2}\right)W = 0, \tag{63}$$

Our derivation holds in the limit that $gH/c_s^2 \ll 1$. The student interested in a more complete derivation which does not make this assumption may want to take a look at Glenn's notes on the Tidal Laplace Equations posted on the class website.